I. The Syzygetic Pencil of Cubics and a New Geometrical Development of its Hesse Group G_{216}.
II. On the Complete Pappus Hexagon.

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Introduction.

1. The course of lectures by Prof. F. Morley during the winter of 1903-4 on cubic curves suggested this dissertation and prepared me to carry on the research. The trend of the work was largely determined by an incidental question by Prof. A. Cohen as to the groups involved in the system of conics I had just presented to the Mathematical Seminary. The interest and valuable suggestions of Dr. A. B. Coble in the carrying on of the work are gratefully acknowledged.

2. The close connection between the Hesse group and the syzygetic pencil of cubics makes it necessary to say something at least about this pencil of curves. Without attempting even an outline of the theory, I present in Chapter I. only such matter as is needed later, besides some new facts concerning the pencil and a figure showing the appearance

or some noteworthy and specially related cubics of the Pencil. No figure seems ever to have been published except that in connection with the paper of Prof. Morley in Proceedings of the London Math. Society, Ser. 2, Vol. 2, Part 2, which shows arbitrarily selected ~~cubics~~. The first work leading to that figure was done by me. Therefore I present the figure here, also one of the corresponding polar-reciprocal range of line-cubics.

Chapter II. shows how to derive a closed system of thirty-six conics analogous to the conic of Chapter I. as to which the syzygetic pencil and range are polar reciprocal. It also discusses the action of their polarities upon the four inflexional triangles and presents some history of similar considerations.

In Chapter III. there is given a brief outline of the history of attempts to determine all finite groups of transformations, and in particular an account of the Hesse Group of 216 Collineations. Further, in

arise and write down the matrices of these collineations by means of the closed system of thirty-six conics, which are here differently defined and given from their new definition. All the sub-groups are found and discussed and the collineations are classified as to periodicity.

Chapter IV. treats of perspective triangles in other perspective forms than six-fold as are those previously considered. Triply perspective triangles are especially considered. As a second way to secure triangles in three-fold perspective also some in two-fold and single perspective to develop what we call the Complete Pappus Hexagon, and deduce a number of theorems connected with it.

Chapter I.

The Syzygetic Pencil of Cubics.

3. The name syzygetic [1] is given to the pencil of cubics determined by a cubic f and its Hessian Δ,

$$\kappa f + \lambda \Delta = 0. \qquad (1)$$

It is the pencil of cubics having the same nine points of inflexion, and consequently the same four inflexional triangles. By varying λ we get all cubics of the pencil.

It is well known that any non-singular cubic may be brought in four ways, as shown for example by Weber [2], into Hesse's canonical form $x_1^3 + x_2^3 + x_3^3 + 6m\, x_1 x_2 x_3 = 0$. (2)

This simply means that the cubic has been referred to one of its inflexional triangles as reference triangle. The Hessian covariant

(1) Clebsch-Lindeman: Leçons sur la Géométrie, II., p. 230.

(2) Lehrbuch der Algebra, 2 Aufl. II., §§106, 107; pp. 399-401.

See also C-L., II, p. 2..

of the form (2) is

$$\Delta \equiv -m^2(x_1^3+x_2^3+x_3^3)+(1+2m^3)x_1x_2x_3 \qquad (3)$$

Its vanishing gives the Hesse Cubic or the Hessian. Thus, if $\underline{m}$ is the parameter of the cubic (2) and $\underline{m'}$ is that of its Hessian, $6m' = -\frac{1+2m^3}{m^2}$; and we see that the Hessian is also a curve of the pencil of cubics given by (2) for all values of $\underline{m}$ from $+\infty$ to $-\infty$.

The relation between $\underline{m}$ and $\underline{m'}$ further shows that for each value of $\underline{m'}$ there are three values of $\underline{m}$, but for each value of $\underline{m}$ there is but one value of $\underline{m'}$; or in other words, each cubic of the pencil is Hessian of three cubics while it has but one Hessian.

Since the Hessians are likewise given by (2) when $\underline{m}$ is parameter, equation (2) as well as equation (1) gives the syzygetic pencil. In the form (2) we shall consider it.

4. The nine inflexions lie by threes on [illegible]

sides of each of the inflexional triangles, and so on the sides of the one we chose as triangle of reference with vertices whose coördinates are respectively $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. The coördinates of these nine inflexions are found therefore as the intersections of the successive sides with the cubic, by placing successively $x_1 = 0$ and $x_2 = 1$, $x_2 = 0$ and $x_3 = 1$, $x_3 = 0$ and $x_1 = 1$, and denoting as we do throughout, the cube roots of unity by $1, \omega, \omega^2$:

$$\begin{array}{lll} (0, 1, -1), & (0, 1, -\omega), & (0, 1, -\omega^2), \\ (-1, 0, 1), & (-\omega, 0, 1), & (-\omega^2, 0, 1), \\ (1, -1, 0), & (1, -\omega, 0), & (1, -\omega^2, 0) \end{array} \qquad (4)$$

5. Speaking of this array as the terms of a three-row determinant, we see that these nine points lie on four triangles, the sides of which are determined by and contain respectively (¹),

A: the points of the rows separately.

B: the points of the columns separately.

(¹) C-L, II., (7) p. 233.

C: the points of the three positive diagonals.

D: the points of the three negative diagonals.

The sides of these triangles have the following equations respectively:

A.	B.	C.	D.	
$x_1 = 0.$	$x_1 + x_2 + x_3 = 0.$	$\omega^2 x_1 + x_2 + x_3 = 0.$	$\omega x_1 + x_2 + x_3 = 0.$	(5)
$x_2 = 0.$	$x_1 + \omega^2 x_2 + \omega x_3 = 0.$	$x_1 + \omega^2 x_2 + x_3 = 0.$	$x_1 + \omega x_2 + x_3 = 0.$	
$x_3 = 0.$	$x_1 + \omega x_2 + \omega^2 x_3 = 0.$	$x_1 + x_2 + \omega^2 x_3 = 0.$	$x_1 + x_2 + \omega x_3 = 0.$	

The vertices opposite these respective sides are,

$\xi_1 = 0.$	$\xi_1 + \xi_2 + \xi_3 = 0.$	$\omega \xi_1 + \xi_2 + \xi_3 = 0.$	$\omega^2 \xi_1 + \xi_2 + \xi_3 = 0.$	
$\xi_2 = 0.$	$\xi_1 + \omega \xi_2 + \omega^2 \xi_3 = 0.$	$\xi_1 + \omega \xi_2 + \xi_3 = 0.$	$\xi_1 + \omega^2 \xi_2 + \xi_3 = 0.$	(6)
$\xi_3 = 0.$	$\xi_1 + \omega^2 \xi_2 + \omega \xi_3 = 0.$	$\xi_1 + \xi_2 + \omega \xi_3 = 0.$	$\xi_1 + \xi_2 + \omega^2 \xi_3 = 0.$	

6. The first polar of any point y as to a cubic curve is of course a conic. When this point y is an inflexion of the cubic, the polar conic breaks up into two linear factors (¹), the inflexional tangent and harmonic polar of the point. Performing the polar operation as to the cubics of the pencil (2) for the various inflexions we obtain the following results,—

(¹) C.-L., II., p. 22.

(¹) Salmon: Higher Plane Curves 3rd Ed., §§ 74, 171, pp. [illegible]

Coörds. of Inflexions.	Respective Polar Conics.	Har. Polars.	Inflexional Tangents.
$0,\ 1,\ -1.$	$x_2^2-x_3^2-2mx_1(x_2-x_3)=0.$	$x_2=x_3.$	$x_2+x_3-2mx_1=0.$
$-1,\ 0,\ 1.$	$x_3^2-x_1^2-2mx_2(x_3-x_1)=0.$	$x_3=x_1.$	$x_3+x_1-2mx_2=0.$
$1,\ -1,\ 0.$	$x_1^2-x_2^2-2mx_3(x_1-x_2)=0.$	$x_1=x_2.$	$x_1+x_2-2mx_3=0.$
$0,\ 1,\ -\omega.$	$\omega^2x_2^2-x_3^2-2mx_1(x_2-\omega^2x_3)=0.$	$x_2=\omega^2x_3.$	$x_2+\omega^2x_3-2\omega mx_1=0.$
$-\omega,\ 0,\ 1.$	$\omega^2x_3^2-x_1^2-2mx_2(x_3-\omega^2x_1)=0.$	$x_3=\omega^2x_1.$	$x_3+\omega^2x_1-2\omega mx_2=0.$
$1,\ -\omega,\ 0.$	$\omega^2x_1^2-x_2^2-2mx_3(x_1-\omega^2x_2)=0.$	$x_1=\omega^2x_2.$	$x_1+\omega^2x_2-2\omega mx_3=0.$
$0,\ 1,\ -\omega^2.$	$\omega x_2^2-x_3^2-2mx_1(x_2-\omega x_3)=0.$	$x_2=\omega x_3.$	$x_2+\omega x_3-2\omega^2mx_1=0.$
$-\omega^2,\ 0,\ 1.$	$\omega x_3^2-x_1^2-2mx_2(x_3-\omega x_1)=0.$	$x_3=\omega x_1.$	$x_3+\omega x_1-2\omega^2mx_2=0.$
$1,\ -\omega^2,\ 0.$	$\omega x_1^2-x_2^2-2mx_3(x_1-\omega x_2)=0.$	$x_1=\omega x_2.$	$x_1+\omega x_2-2\omega^2mx_3=0.$

(7)

The harmonic polars are independent of the parameter but the inflexional tangents are not, so there are but nine harmonic polars for the pencil and an inflexional tangent at each inflexion for each cubic of the pencil.

7. Some particular cubics of the pencil.

(a) The cubics of the pencil, whose parameter m is resp. $\infty, -\frac{1}{2}, -\frac{\omega^2}{2}, -\frac{\omega}{2}$, are

$$A:\ x_1x_2x_3=0, \qquad C:\ x_1^3+x_2^3+x_3^3-3\omega^2x_1x_2x_3=0, \tag{8}$$

$$B:\ x_1^3+x_2^3+x_3^3-3x_1x_2x_3=0, \qquad D:\ x_1^3+x_2^3+x_3^3-3\omega x_1x_2x_3=0,$$

which are as lettered resp. the four inflexional triangles of equations (5), two real and two imaginary.(1)

(1) C.-L., II., p. 230, also 239, 240.

The polar line (¹) as to these triangles of any point y has coördinates respectively,

	x_1	x_2	x_3
A:	$y_2 y_3$	$y_3 y_1$	$y_1 y_2$
B:	$y_1^2 - y_2 y_3$	$y_2^2 - y_3 y_1$	$y_3^2 - y_1 y_2$
C:	$y_1^2 - \omega^2 y_2 y_3$	$y_2^2 - \omega y_3 y_1$	$y_3^2 - \omega^2 y_1 y_2$
D:	$y_1^2 - \omega y_2 y_3$	$y_2^2 - \omega y_3 y_1$	$y_3^2 - \omega y_1 y_2$

The determinant formed from any three rows of these coefficients vanishes identically, therefore the four polars pass through a common point, so we say

<u>The four polar lines of any point as to the four inflexional triangles meet in a point</u>, or also

<u>Any two of the inflexional triangles are apolar as seen from any point of the plane, and are thus syzygetic.</u>

(b) The first of two covariant cubics of the pencil (²) is the Hessian, whose equation is (3) p. 8. It cuts out the singular points of its cubic, for it is the locus of points whose polar conics break up into two right

(¹) Salmon: H.P.C., §165, p. 143. (²) C.-L., II., p. 7.

lines, and so with its cubic determines the pencil.

(d) The Cayleyan of the cubic $x_1^3+x_2^3+x_3^3+6mx_1x_2x_3=0$.
The polar conic as to this cubic of a point y is

$$y_1(x_1^2+2mx_2x_3)+y_2(x_2^2+2mx_3x_1)+y_3(x_3^2+2mx_1x_2)=0. \qquad (9)$$

Considering the y's as parameters, this is a net of polar conics. By inspection we see that the conic $m\xi_1^2-\xi_2\xi_3$ is apolar with the three conics of (9), likewise the conics $m\xi_2^2-\xi_3\xi_1$, $m\xi_3^2-\xi_1\xi_2$. So the net has a corresponding web of apolar conics, given by the equation

$$\eta_1(m\xi_1^2-\xi_2\xi_3)+\eta_2(m\xi_2^2-\xi_3\xi_1)+\eta_3(m\xi_3^2-\xi_1\xi_2)=0, \qquad (10)$$

where the η's are parameters.

The contravariant or Jacobian of the web is the Cayleyan

$$\begin{vmatrix} 2m\xi_1 & -\xi_3 & -\xi_2 \\ -\xi_3 & 2m\xi_2 & -\xi_1 \\ -\xi_2 & -\xi_1 & 2m\xi_3 \end{vmatrix} = 0$$

or $$m(\xi_1^3+\xi_2^3+\xi_3^3)+(1-4m^3)\xi_1\xi_2\xi_3=0, \qquad (11)$$

which is a cubic of the range (1) enveloped by the ∞^2 lines composing the degenerate conics, the

(1) C.-L., II., p. 244.

first polars of points along the Hessian as to the cubic.

Here, as in the case of the Hessian, we see that each curve of the pencil has but one Cayleyan while it is the Cayleyan of three cubics of the pencil.

(d) The simplest invariant is found by operating with the Cayleyan (11) on the cubic (2). It is the quartic invariant denoted in Salmon by **S**. By operating as stated, we have after dividing by 24[1],

$$S = m(1-m^3). \qquad (12)$$

The vanishing of this invariant gives four curves of the pencil for which $m = 0, 1, \omega^2, \omega$ respectively, the so called equianharmonic cubics:

$$S_1:\ x_1^3+x_2^3+x_3^3=0. \qquad S_3:\ x_1^3+x_2^3+x_3^3+6\omega^2 x_1x_2x_3=0. \qquad (13)$$
$$S_2:\ x_1^3+x_2^3+x_3^3+6x_1x_2x_3=0. \qquad S_4:\ x_1^3+x_2^3+x_3^3+6\omega x_1x_2x_3=0.$$

This name is given to these cubics because the anharmonic ratio of the four tangents [2] drawn from a point of the curve to itself, which is constant, is equianharmonic in these four cases.

[2] C.-L., II., pp. 325, 326. [1] Salmon: H.P.C., §220, p. 191

(c) By operating with the Cayleyan (11) on the Hessian (3) we have the sextic invariant, **T** of Salmon [1]. It is

$$T = 1 - 20m^3 - 8m^6. \qquad (14)$$

The curves given for the vanishing of **T** are the six harmonic cubics of the pencil, called so because the anharmonic ratio of the four tangents from any point on the curve is harmonic [2].

The Drawing of the Syzygetic Pencil.

8. As a far more convenient form of the cubics of the pencil for purposes of construction, we transform so as to have inflexions at the circular imaginary points **I** and **J**, by putting

$$x_1 = x + y + 1, \quad x_2 = -(x-1), \quad x_3 = -(y-1). \qquad (15)$$

By this transformation, the equation (2) of the pencil becomes

$$xy(x+y) + \frac{2(1-m)}{1+2m}(x^2+xy+y^2) + 1 = 0,$$

or it is of the form —

(1) Salmon: H.P.C., §221, p.173, also Weber II., s. 406, (7). (2) C.-L., II., p. 326.

$$xy(x+y)+\mu(x^2+xy+y^2)+1=0, \qquad (16)$$

where $$\mu=\frac{2(1-m)}{1+2m}, \text{ and } m=\frac{2-\mu}{2(1+\mu)}. \qquad (17)$$

9. The special cubics of §7.

(a) The four inflexional triangles (8) of which triangle A was the reference triangle become in these coördinates

$$A: \quad (x+y+1)(x-1)(y-1)=0,$$
$$B: \quad x^2+xy+y^2=(x-\omega y)(x-\omega^2 y)=0,$$
$$C: \quad xy(x+y)-\omega(x^2+xy+y^2)+1=0,$$
$$D: \quad xy(x+y)-\omega^2(x^2+xy+y^2)+1=0, \qquad (18)$$

for from (17) $\mu=-1, \infty, -\omega, -\omega^2$ respectively.

Our former reference triangle is seen thus to be now the line through the points ω and ω^2 on the unit-circle, and the point $x=1$, on the real axis, taken twice. This point is according to Salmon (1) an <u>acnode</u> or <u>conjugate point</u>. The other two sides of the triangle A are the circular imaginary rays through this point.

(1) H.P.C., §38, II., p. 25.

Fig. 1.

Full lines are degenerate cubics of the pencil.
Dash lines are the three real harmonic polars.

Since the cube terms are lacking in **B**, the cubic consists of the line at infinity and the lines $x-\omega y=0$ and $x-\omega^2 y=0$, which pass through the origin and the points ω^2 and ω respectively.

The representable parts are given in the accompanying figure.

(b) The Hessian is by direct calculation on (16)

$$H \equiv \frac{1}{6}\begin{vmatrix} 2y+2\mu & 2(x+y)+\mu & \mu(2x+y) \\ 2(x+y)+\mu & 2x+2\mu & \mu(2y+x) \\ \mu(2x+y) & \mu(2y+x) & 6 \end{vmatrix}$$

or $H \equiv 3\mu^2 xy(x+y)-(\mu^3+4)(x^2+xy+y^2)+3\mu^2=0,$ (19)

which is of the form of the cubic (16) with its parameter μ' given by

$$\mu' = -\frac{\mu^3+4}{3\mu^2}. \qquad (20)$$

Therefore, equation (16) likewise gives the syzygetic pencil for all values of μ from $+\infty$ to $-\infty$ as well as equation (1).

(c) The Cayleyan.

The polar conic as to the cubic (16) of a point p

is $p_1[2xy+y^2+\mu(2x+y)]+p_2[x^2+2xy+\mu(x+2y)]+p_3[\mu(x^2+xy+y^2)+3]=0$ (21)

Regarding the p's as parameters this is the <u>net of polar conics</u>.

To find the <u>web of apolar conics</u> we take the general line-conic $a\xi^2+b\eta^2+c\zeta^2+d\xi\eta+e\eta\zeta+f\zeta\xi=0$, and operate separately on the point conics of (21). Thus we obtain

$$\begin{aligned} 2b+2d+2\mu f+\mu e&=0,\\ 2a+2d+\mu f+2\mu e&=0,\\ 2\mu a+2\mu b+\mu d+6e&=0. \end{aligned} \qquad (22)$$

We desire apolar conics analogous in form respectively to those of equation (21) and so we first take that $a=0$, $b=d=1$, whence, by substituting in equations (22) and in the general line-conic above, we get the conic $2\mu\xi\eta+2\mu\eta^2-4\xi-\mu^2=0$.

Second, put $b=0$, $a=d=1$, and get the conic

$$2\mu\xi^2+2\mu\xi\eta-4\eta-\mu^2=0.$$

Third, put $a=1$, $d=f=0$, and the corresponding conic is

$$2(\xi^2-\xi\eta+\eta^2)-\mu=0.$$

By these we may write the apolar web of line conics,

$$\pi_1[2\mu\xi\eta+2\mu\eta^2-4\xi-\mu^2]+\pi_2[2\mu\xi^2+2\mu\xi\eta-4\eta-\mu^2]+\pi_3[2(\xi^2-\xi\eta+\eta^2)-\mu]=0. \quad (23)$$

The Jacobian of this net (23) is directly calculated to be

$$\begin{vmatrix} \mu\eta-2 & \mu(\xi+2\eta) & -(2\xi+\mu^2) \\ \mu(2\xi+\eta) & \mu\xi-2 & -(2\eta+\mu^2) \\ 2\xi-\eta & -\xi+2\eta & -\mu \end{vmatrix}$$

or $2\mu(\xi^3+\eta^3)-3\mu\xi\eta(\xi+\eta)+(\mu^3-2)(\xi^2-\xi\eta+\eta^2)-\mu^2(\xi-\eta)-\mu=0$, (24)

which is the Cayleyan of the cubic (16).

(d) By operating with the Cayleyan (24) on the cubic (16) we have the quartic invariant

$$S=\mu^4-8\mu, \qquad (25)$$

whose vanishing gives four cubics of the pencil, viz., those with parameters $\mu=0, 2, 2\omega, 2\omega^2$. The two real ones of these will be drawn on our figure.

(e) The Cayleyan operating on the Hessian (19) gives the sextic invariant

$$T=8-20\mu^3-\mu^6, \qquad (26)$$

whose vanishing gives six more cubics of the pencil with parameters given by $\mu^3=-10\pm6\sqrt{3}$; i.e., three sets of two each. For the [illegible] signs.

there are in each case one real and two imaginary roots. The two real roots are $-(1-\sqrt{3})$ and $-(1+\sqrt{3})$. The [illegible] harmonic cubics are [illegible]

10. The nine inflexions and harmonic polars of equations (7) p. 11, are, in order there given, in conjugate coördinates as follows, with the position on the figure as indicated respectively,—

Inflexion.	Point on fig.	Its harmonic polar.	The line on fig.
1	At ∞ on line $\overline{\omega\omega^2}$.	$x-y=0.$	Axis of reals.
2	I	$x+2y=0.$	Imaginary circular rays from origin.
3	J	$2x+y=0.$	
4	ω^2	$x-\omega^2 y+\omega^2-1=0.$	Line $\overline{1,-\omega^2}$.
5		$x-\omega^2 y-\omega+1=0.$	An imag. cir. ray from ω.
6		$x-\omega^2 y-\omega^2+\omega=0.$	" " " " " ω^2.
7	ω	$x-\omega y+\omega-1=0.$	Line $\overline{1,-\omega}$.
8		$x-\omega y-\omega^2+1=0.$	An imag. cir. ray from ω^2.
9		$x-\omega y-\omega+\omega^2=0.$	" " " " " ω.

(27)

By the projection of the pencil into this form we see we have the line at infinity as one real side of an inflexional triangle, and we have one real inflexion at infinity. That leaves (¹) to appear on figure (opposite page 31.) three sides of inflexional triangles, two inflexions, and three harmonic polars.

(¹) C.-L., II., p. 235, on the real parts of the figure. Cf. equations [illegible]

11. The Asymptotes.

Join the line $x+y=\lambda$, which is perpendicular to the axis of reals at distance $\frac{1}{2}\lambda$ from the origin; that is, the line in which the reflexion of the origin is the point λ. It cuts the cubic (16) where

$$\lambda x(\lambda-x)+\mu(x^2-\lambda x+\lambda^2)+1=0.$$

Since the cube terms vanish, the line meets the cubic at infinity. If next we put $\lambda=\mu$, we have only

$$\mu^3+1=0.$$

Hence, since the square terms also vanish, the line $x+y=\mu$ is tangent to the cubic at infinity, in direction perpendicular to the axis of reals, and is the asymptote. It is an inflexional tangent as well.

Thus when μ is given, the asymptote of the cubic of the pencil for that particular μ as parameter is also given as the perpendicular to the axis of reals at the distance $\frac{1}{2}\mu$ from the origin.

12. By taking the first polar of the infin–

ω as to the general cubic (16) we find it breaks up into the two linear factors

$$\{x-\omega y+\omega-1\}\{\omega^2(x-\omega y-\omega+1)+\mu(x-\omega^2 y)\}=0,$$

the harmonic polar and the flex-tangent respectively.

The latter cuts the asymptote of the cubic, $x+y=\mu$, where $-(1+\mu)(\omega x+\omega^2\mu-\omega^2+1)=0$.

The harmonic polar of ω^2, equation (27), cuts the asymptote where $-(\omega x+\omega^2\mu-\omega^2+1)=0$. Therefore, we may draw the flex-tangents to any cubic of the pencil by drawing from either inflexion, ω or ω^2, to the point where the harmonic polar of the other cuts the asymptote of the curve.

13. Intersections of the cubics with the real axis

The general cubic (16) cuts this axis, $x-y=0$, where

$$2x^3+3\mu x^2+1=0. \qquad (28)$$

The discriminant (*) of this equation is

$$D=-108(\mu^3+1), \qquad (29)$$

which shows that if

*Weber: Bd. I., S. 273

1°. $\mu < -1$, then $D < 0$, and the equation has three, real, distinct roots.
2°. $\mu = -1$, then $D = 0$, " " " " three, real, roots, one repeated.
3°. $\mu > -1$, " $D > 0$, " " " " one real, two imag. roots.

As to the cubic this says

1°. that for $\mu < -1$, the cubic cuts the axis of reals in three points and hence is, in general, of the bipartite type.

2°. that for $\mu = -1$, there is one actual intersection and an acnode on the axis, as we saw in §9 (a'), p. 16.

3°. that for $\mu > -1$, there is but one real intersection and the cubic is of the unipartite type.

The study of these intersections thus enables us to classify the cubics of the pencil from the parameter, and we may say in conclusion that $\mu = -1$ gives a transition cubic whose oval becomes a point and whose serpentine branch is a right line, the asymptote of the curve itself.

For μ having values from -1 to $-\infty$, the serpentine branch bulges slightly towards the origin

as the number increase in absolute value, and the oval increases <u>rapidly</u> at the same time, until, when $\mu=-\infty$, the two branches unite at the origin and become two straight lines together with the line at infinity perpendicular to the axis of reals. This is the second degenerate cubic of the pencil. For values of μ from -1 to $+\infty$ the cubics are unipartite, the "bay" increasing with increasing values of μ, bulging <u>from</u> the origin towards the left.

14. The plotting of the curves is much facilitated by drawing a number of circles concentric with the unit-circle and calculating the intersections to the cubics, also by drawing the flex-tangents at ω and ω^2 as shown in §12, p. 22.

The cubic (16), $xy(x+y)+\mu(x^2+xy+y^2)+1=0$, cuts the circle $xy=\rho^2$ where

$$\mu\left(x+\frac{\rho^2}{x}\right)^2+\rho^2\left(x+\frac{\rho^2}{x}\right)+1-\rho^2\mu=0.$$

Since the x and y are conjugate complex coördinates, $x+\frac{\rho^2}{x}=c$ is a right line perpendicular

to the axis of reals at a distance $\frac{1}{2}c$ from the origin. Therefore, the cubic cuts the circle where this perpendicular does for values c which are the roots of the equation in $(x+\frac{\rho^2}{x})$. The location of the perpendicular is given by

$$x = \frac{-\rho^2 \pm \sqrt{4\mu(\rho^2\mu-1)+\rho^4}}{[illegible]} \qquad (30)$$

15. The Hessian cubic whose parameter is μ' has three cubics of which it is the Hessian. Their parameters are the roots of equation (20).

$$\mu^3+3\mu'\mu^2+4=0.$$

Their asymptotes cut the axis of reals at $x=\frac{1}{2}\mu$ resp. The Hessian cubic itself cuts the axis of reals at points given by the roots of equation (28),

$$2x^3+3\mu'x^2+1=0.$$

If we put in this $x=\frac{1}{2}\mu$, we have identically equation (20). Therefore, the Hessian cubic cuts the axis where the asymptotes of its three cubics do, and these asymptotes are its tangents at its intersections with the axis of reals.

Or, for this form of the pencil, we may say,

The Hessian cubic is tangent on the axis of reals to its curves' asymptotes.

Since the axis of reals is an harmonic polar and the asymptotes are as well inflexional tangents as seen before, we may state the theorem projectively.

"*The three flex-tangents of the three cubics with a common Hessian, at any one inflexion, touch this Hessian in three points on the harmonic polar of the inflexion considered.*"(')

The discriminant of equation (20) on previous page is, $D = -432(\mu'^3+1)$.

Thus, if the *Hessian cubic* is *bipartite*, i.e., $\mu' < -1$, there are *three real unipartite cubics* of which it is the Hessian; for the roots of equation (20) are then all real, two positive, one negative but > -1.

If $\mu' = -1$, it is the Hessian of cubics whose parameters are $-1, 2, 2$, that is, Hessian of itself and also of one of the two real equianharmonic

(') This is an extension of theorems by
Alfred Clebsch: Ueber die Wendetangenten der Curven 3ter Ord.; Crelle Jour. 58, S. 232.
Peter Muth: Ueber ternäre Formen u.s.w.; Inaug. Diss. – Giessen 1890, S. 16.

taken twice.

If the Hessian is unipartite, i.e., $\mu' > -1$, it is the Hessian of one real, bipartite cubic. The other real syzygetic harmonic cubic than the one just mentioned, is Hessian of one bipartite cubic with parameter $-\sqrt[3]{4}$; i.e., of one real and two imaginary cubics.

The special theorem at the top of the previous page likewise involves these facts, for a unipartite cubic cuts the axis of reals in one real and two imaginary points and a bipartite cubic cuts in three real points.

Since our names unipartite and bipartite, following Salmon, are actually names of the cubics von Staudt calls resp. odd- and even-circuit cubics, and since also (1) by no projection does a non-singular cubic change its class of odd- or even-circuit, these facts as to the Hessian and its cubics remain for all real projections.

(1) C.-L. II., end of foot-note, p. 223.

16. The two real, [illegible] harmonic cubics are readily seen to be mutually Hessian and cubic, for the parameters $-(1+\sqrt{3})$ and $-(1-\sqrt{3})$ may be interchangeably μ and μ' and satisfy equation (20), $\mu^3 + 3\mu'\mu^2 + 4 = 0$.

Therefore, the flex-tangents of each touches the other and so, what is the same, each cuts the axis of reals at the asymptote of the other for our form of the pencil. Thus, if a square be constructed with the chord $\omega\omega^2$ of the unit-circle as diagonal, these two cubics pass, one through each of the other two vertices of the square while its asymptote passes through the opposite vertex.[1]

17. For the particular form of the pencil under consideration, the two invariants are seen very nicely as the invariants of the quartic[2] giving the intersections of the four tangents from the inclusion at infinity perpendicular to the axis of reals.

As seen in §11 the asymptote at flex-tangent

[2] Salmon: H.P.C., §228, p. 199. [1] Cf. Crelle Jour. Bd. 58, [illegible]

at infinity cuts the axis at $x=\frac{1}{2}\mu$, and equation (28) shews that the cubic cuts this axis orthogonally when

$$2x^3+3\mu x^2+1=0.$$

Therefore, the intersections of the four tangents from the real inflexion at infinity are given by the quartic

$$(2x^3+3\mu x^2+1)(2x-\mu)=0$$

or

$$4x^4+4\mu x^3-3\mu^2x^2+2x-\mu=0.$$

The invariants (1) of this quartic are those of the cubic, given in (25) and (26), to within a numerical factor.

The Syzygetic Range of Cubics. (2)

18. From §5 with its equations we see the invariant parts of the pencil are

9 inflexions which lie by threes on

12 inflexional lines forming the

4 inflexional triangles whose vertices are

12 points, 3 each on the sides of these triangles. They are the Hessian pairs of the three inflexions on their resp. lines. Also there are

9 harmonic polars which with the three tangents

(1) Weber: I. §70, s. 230; II., §108, s. 406. (2) C.-L., II., pp. 241 of

constitute the joint polars of the nine inflexions as to the cubics of the pencil. These nine polars lie by threes on the 12 points above mentioned, and they in turn lie by fours on the 9 harmonic polars.

The systems of 9 inflexions and 9 harmonic polars and of 12 vertices and 12 sides are dualistic and correspond, the 9 points with their 9 lines, the 12 vertices with the 12 sides respectively opposite.

19. In view of this dualistic correspondence, we observe that the polarity arising from the conic $x_1^2+x_2^2+x_3^2=0$, (31)

namely, $x_1=\xi_1$, $x_2=\xi_2$, $x_3=\xi_3$,

sends each of the parts into its corresponding part. At the same time, it sends the point-cubics (of order 3) of the syzygetic pencil into the line-cubics (of class 3) of the syzygetic range

$$\xi_1^3+\xi_2^3+\xi_3^3+6m\xi_1\xi_2\xi_3=0. \qquad (32)$$

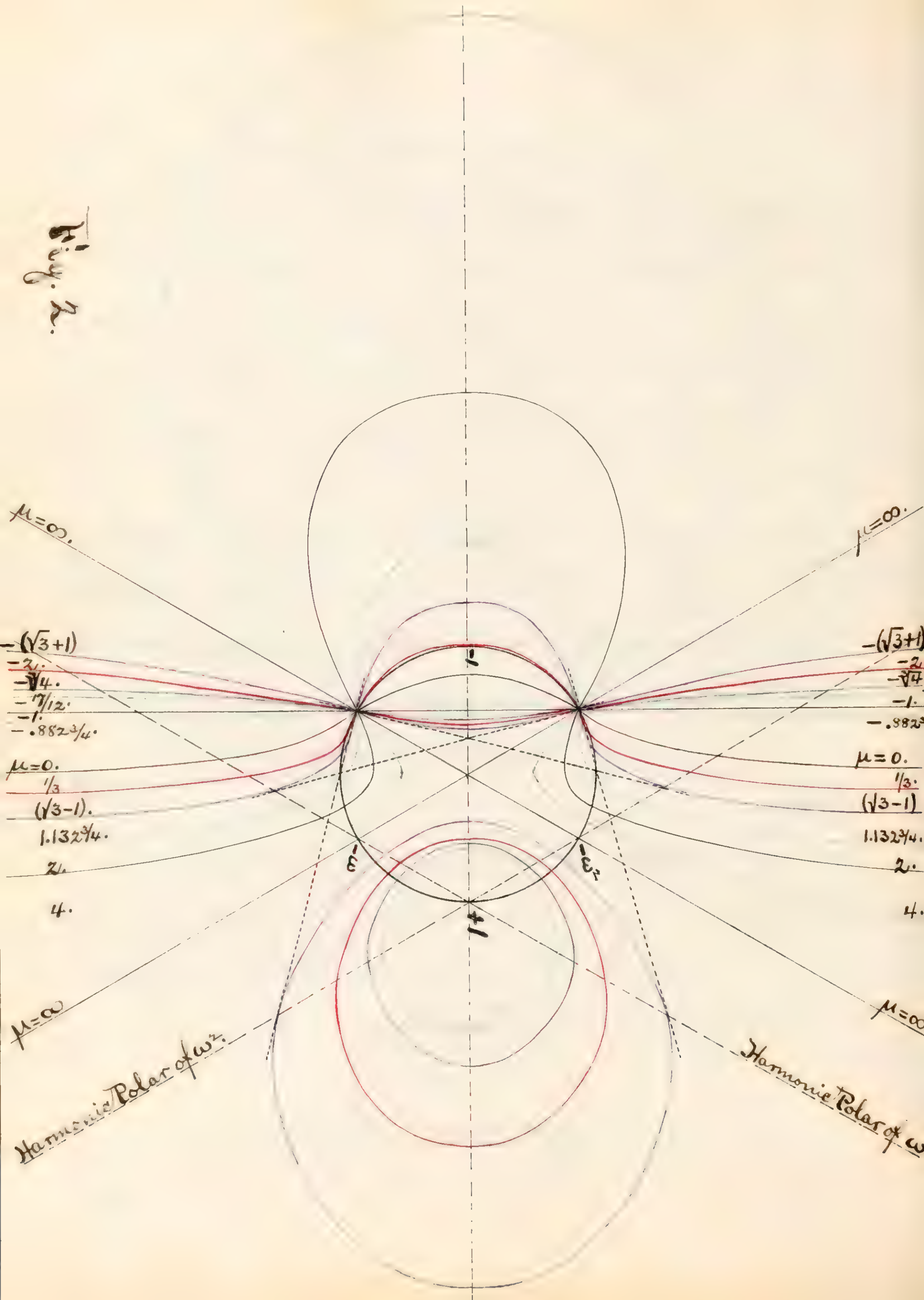
The conic (31) transformed into conjugate coordinates becomes $2(x^2+xy+y^2)+3=0$, (33)

which equation shows the conic, nulliparlite for real reference frame, to be, as to the partly imaginary frame chosen in §§ (15), an hyperbola with the inflexional lines, B (18), $x - \omega y = 0$ and $x - \omega^2 y = 0$ as asymptotes, and with vertices at $\pm \frac{i}{2}\sqrt{6}$; or, transformed to rectangular coördinates with same origin, it is $X^2 - 2Y^2 + 3 = 0$; with the asymptotes as axes, it is $XY = -\frac{3}{2}$. From this latter equation we easily construct the conic, using the fact that, the segments of any chord contained between the curve and its asymptotes are equal.

20. By the polar, reciprocal process as to this conic we deduce the line-cubics, the polar-reciprocals of the point-cubics given in Fig. 2, and thus form Fig. 3 the syzygetic range. The harmonic polars are seen to become the cusp-tangents common to all the curves of the range.

Compare Clebsch's drawing of the syzygetic range [C.-L.: II., p. 243] and note its ideal form. He does not construct it.

Fig. 2.



μ=∞.
μ=∞.
-(√3+1)
-2.
-∛4.
-17/12.
-1.
-.882¾.
μ=0.
1/3
(√3-1).
1.132¾.
2.
4.
μ=∞
-(√3+1)
-2.
-∛4
-1.
-.882¾
μ=0.
1/3.
(√3-1)
1.132¾.
2.
4.
μ=∞
1
ε
ε²
Harmonic Polar of ω².
Harmonic Polar of ω

Explanation of the Figures.

The cubics of the pencil.

The equianharmonic cubic for which $\mu = 0$ cuts the real axis at $x = -0.79$; the one for which $\mu = 2$ cuts the real axis at $x = -3.05$. These are drawn in black.

The two real harmonic cubics for which $\mu = \sqrt{3} - 1$ and $\mu = -(\sqrt{3} + 1)$ cut the axis, as shown in §16, p. 28, at the points $x = -(\sqrt{3}+1)$ and $x = \sqrt{3} - 1$, respectively. The former is unipartite, the latter bipartite. They are drawn in purple and have their inflexional tangents drawn from ω and ω^2.

The cubic for which $\mu = -\frac{17}{12}$ is Hessian of three others for which μ is resp. 4, $1.132\frac{3}{4}$, $-.882\frac{3}{4}$. The Hessian cubic, $\mu = -\frac{17}{12}$, cuts the axis at $x = 2$, $.566$; $-.4414$ as follows from the relations between Hessians and their cubics, §15, p. 26. These four cubics are green.

The cubic with $\mu = \frac{1}{3}$ is Hessian of that one

cubic counted three times, for which $\mu=-2$. The cubic $\mu=1/3$ cuts the axis at -1. These two cubics are red.

The equianharmonic cubic, $\mu=0$, is also Hessian of but one real cubic for which $\mu=-\sqrt[3]{4}$. It appears in the figure in blue.

Fig. 3. Most of the above cubics of the pencil were reciprocated in the conic shown on Fig. 3, and the polar reciprocal line-cubics appear in the same colors resp. and are named with the same μ as the corres[illegible]-cubics.

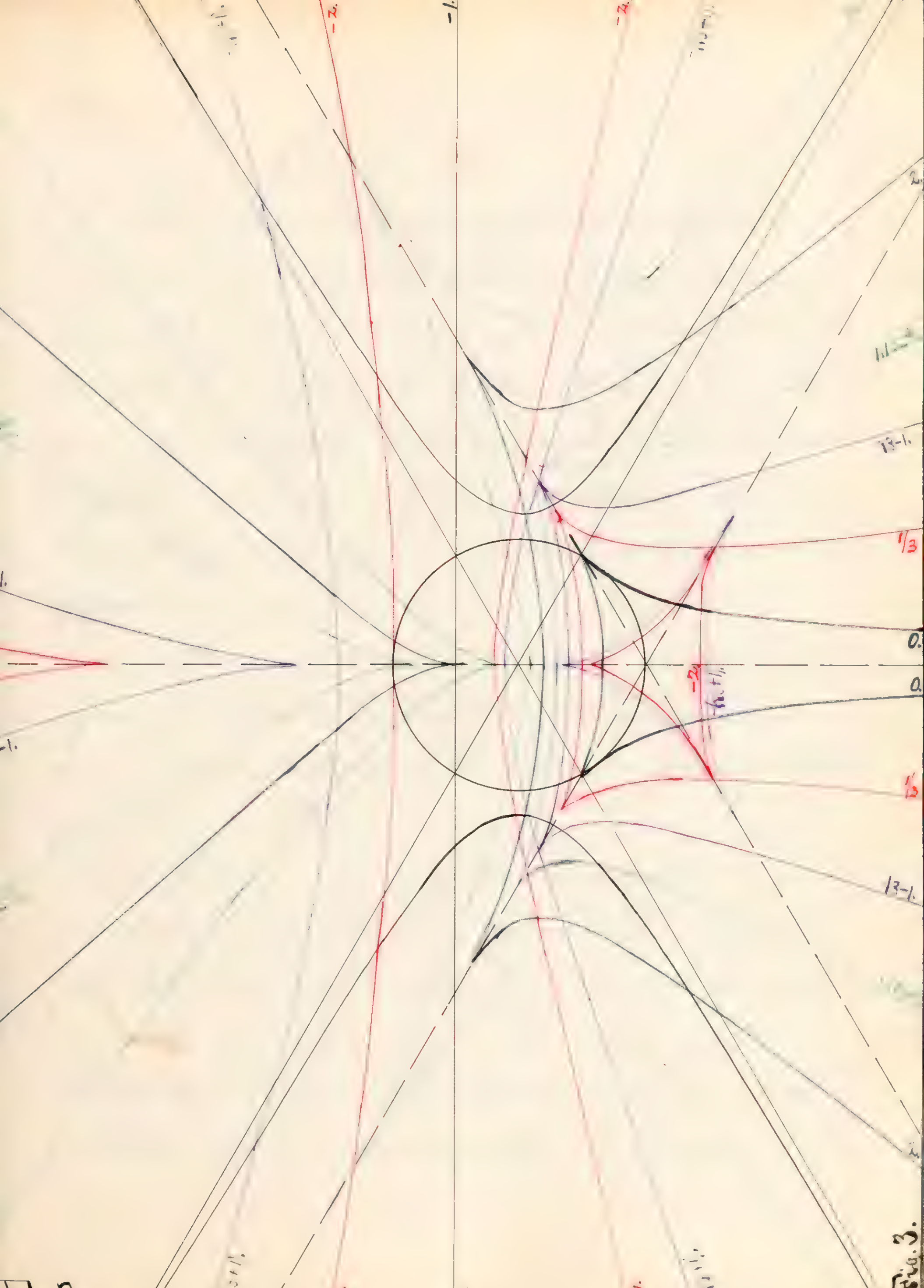


Fig. 3.

Chapter II.

On a Closed System of Thirty-six Conics.

21. The conic (31), given in §19 is referred to triangle A of equations (4) as reference triangle with the first line of triangle B as auxiliary line. This conic it was noted sends the inflexions into corresponding harmonic polars and the vertices of the inflexional triangles respectively into the sides opposite.

The question at once arose as to the effect upon the inflexional triangles of the analogous conics as to all the possible reference frames in the four triangles (4).

The equations of these analogous conics are written by considering separately each of the four triangles as reference triangle with the nine remaining lines in turn as auxiliary lines. Thus, there are thirty-six such conics, nine for each triangle corresponding to the the nine auxili-

ing sides of the triangles (d). As an example, say we wish to consider triangle B in reference triangle with the first side of triangle C as auxiliary line.

Then,

$$X_1' \equiv \omega X_1 + \omega X_2 + \omega X_3 = 0,$$

$$X_2' \equiv X_1 + \omega^2 X_2 + \omega X_3 = 0,$$

$$X_3' \equiv X_1 + \omega X_2 + \omega^2 X_3 = 0,$$

hence, $X_1' + X_2' + X_3' =$ [illegible] $(\omega^2 X_1 + X_2 + X_3)$ = 0 is auxiliary line as we desired. The conic $X_1'^2 + X_2'^2 + X_3'^2 = 0$ has therefore as to the original reference frame the equation $\omega X_1^2 + X_2^2 + X_3^2 + 2(\omega X_2 X_3 + X_3 X_1 + X_1 X_2) = 0$. Similarly by inspection all the others are deduced.

The equations of the conics are withheld here to appear in the next chapter, where they are derived differently and their properties defined. This development in this manner was outlined in The Johns Hopkins University Circular, January 1905, pp. [illegible] [illegible], named the conics by the letter of the reference triangle with a subscript 1 to 9, indicating which of the remaining nine lines of [illegible]

order from the reference triangle is the auxiliary one. Thus the conics are designated by $A_1, A_2, \ldots A_9$, and similarly with B, C, and D.

22. These thirty-six conics form a closed system, for by operating two times with polarity arising from any conic of the system upon any polarity of the system we get a polarity of the system; or the product of three polarities is a polarity of the system.

As to their effect upon the inflexional triangles, the polarities divide into sets in two ways. First, by nines, A_{1-9}, B_{1-9}, etc., they send the vertices respectively of triangles A, B, etc. into the sides opposite, and at the same time reciprocate the other triangles in themselves or one into the other, the vertices into sides in anticyclic order, never however sending the vertices of any triangle part into the sides of one triangle and part into those of another.

Second, they divide into sets of six each, operating

as indicated in the following table:—

Polarities		Triangles			Triangles			
A_{1-3}, B_{7-9}	send	A and B	into themselves respectively, and	C	and	D	into each other.
A_{4-6}, C_{4-6}	"	A " C	"	, "	D	" B	" " "
A_{7-9}, D_{1-3}	"	A " D	"	, "	B	" C	" " "
B_{1-3}, C_{7-9}	"	B " C	"	, "	A	" D	" " "
D_{4-6}, B_{4-6}	"	D " B	"	, "	A	" C	" " "
C_{1-3}, D_{7-9}	"	C " D	"	, "	A	" B	" " "

The classification shows also that besides the nine polarities which send a triangle into itself, vertices into sides opposite, there are nine others sending it into itself, but vertices into sides in anti-cyclic order as to the order of the vertices.

The action of the Polarities on the four triangles is fully presented in the following table:—

The 36 polarities send the vertices 1, 2, 3 of Triangle

Polarity	A resp. into sides	B resp. into sides	C resp. into sides	D resp. into sides
A_1	Opposite.	1,3,2 of B itself.	1,2,3 of D.	1,2,3 of C.
A_2	"	3,2,1 " " "	3,1,2 " "	2,3,1 " "
A_3	"	2,1,3 " " "	2,3,1 " "	3,1,2 " "
A_4	"	1,3,2 " D.	1,3,2 " C itself.	1,3,2 " B.
A_5	"	2,1,3 " "	3,2,1 " " "	2,1,3 " "
A_6	"	3,2,1 " "	2,1,3 " " "	3,2,1 " "
A_7		1,2,3 " C.	1,2,3 " B.	1,3,2 of D itself.
A_8	"	2,3,1 " "	3,1,2 " "	3,2,1 "
A_9	"	3,1,2 " "	2,3,1 " "	2,1,3 " " "
B_1	1,3,2 of D.	Opposite.	1,3,2 " C itself.	1,3,2 " A.
B_2	3,2,1 " "	"	3,2,1 " " "	3,2,1 " "
B_3	2,1,3 " "	"	2,1,3 " " "	2,1,3 " "
B_4	1,3,2 " C.	"	1,3,2 " A.	1,3,2 " D itself.
B_5	3,2,1 " "	"	3,2,1 " "	3,2,1 " " "
B_6	2,1,3 " "	"	2,1,3 " "	2,1,3 " " "
B_7	1,3,2 " A itself.	"	1,3,2 " D.	1,3,2 " C.
B_8	3,2,1 " " "	"	3,2,1 " "	3,2,1 " "
B_9	2,1,3 " " "	"	2,1,3 " "	2,1,3 " "
C_1	1,3,2 of B.	1,3,2 of A.	Opposite.	1,3,2 " D itself.
C_2	2,1,3 " "	2,1,3 " "	"	3,2,1 " " "
C_3	3,2,1 " "	3,2,1 " "	"	2,1,3 " " "
C_4	1,2,3 " D.	1,3,2 " B itself.	"	1,2,3 " A.
C_5	3,1,2 " "	3,2,1 " " "	"	3,1,2 " "
C_6	2,3,1 " "	2,1,3 " " "	"	2,3,1 " "
C_7	1,3,2 " A itself.	1,2,3 " D.	"	1,2,3 " B.
C_8	3,2,1 " " "	2,3,1 " "	"	3,1,2 "
C_9	2,1,3 " " "	3,1,2 " "	"	2,3,1
D_1	1,3,2 " " "	1,3,2, " C.	1,3,2 of B.	Opposite.
D_2	3,2,1 " " "	2,1,3, " "	2,1,3 " "	
D_3	2,1,3 " " "	3,2,1 " "	3,2,1 " "	
D_4	1,2,3 " C.	1,3,2 " B itself.	1,2,3 " A.	
D_5	3,1,2 " "	3,2,1 " " "	2,3,1 " "	
D_6	2,3,1 " "	2,1,3 " " "	3,1,2 " "	
D_7	1,2,3 " B.	1,2,3 " A.	1,3,2 " C itself.	
D_8	3,1,2 " "	2,3,1 " "	3,2,1 " " "	"
D_9	2,3,1 " "	3,1,2 " "	2,1,3 " " "	"

23. It is easily shown that when a conic reciprocates a triangle into another, the two triangles are in perspective, and conversely for two perspective triangles there is a conic which sends one into the other. For two triangles in n-fold perspective there are n such conics. For the four inflexional triangles to be in full or 6-fold perspective two by two mutually, there should be 6 times $_4C_2$ or 36 conics, which there are. The table also shows that the four triangles are mutually in full perspective and so are syzygetic, which fact says by definition that the triangles two by two are seen to be apolar from any point of their plane, or the lines to the vertices of any two are apolar triads.

The projecting rays are the 9 harmonic polars with four vertices of the triangles on each. The centers of perspective are the vertices of the triangles, and the axes their sides. The small seven of the two cyclic and the three anti-cyclic forms of perspective and mean that the vertices are in perspective as indicated.

Cyclic Forms.

1′	2′	3′
1	2	3
2	3	1
3	1	2

Anti-cyclic Forms.

1′	2′	3′
1	3	2
2	1	3
3	2	1

Then we may say, as calculation readily shows, that for any one of the triangles, the vertices and sides of one of the remaining 2 triangles are respectively the centres and axes in cyclic order of the three cyclic perspectivities, and the vertices and sides of the other triangle resp. in anti-cyclic order are the centres and axes of the three anti-cyclic perspectivities.

24. In the Archiv der Mathematik und Physik (1) Aályi has two very neat papers on perspective triangles, the latter of which considers the subject as to conics. He deduces the polar conics and studies their relations and properties.

For 6-fold perspective triangles he finds the six conics and adds, "Unter den 6 Kegelschnitten giebt es höchstens vier reele, die beiden Kreise ...

(1) 1882, Bd. 70, ss. 105-110; and 1884, 2R., II T., ss. [illegible]

inner imaginär." The equations of the conics show that four of the conics at most are real and that one of these is multipartite, but contrary to Lüroth's equations (11) show that one triangle and a side of another may be real. He was led into the error by having taken the sides of two triangles those of triangles **C** and **D**.

The four real conics of the system have the following equations in trilinear coördinates,

$x_1^2+x_2^2+x_3^2=0.$ $\qquad$ $x_2^2+2x_3x_1=0.$

$x_1^2+2x_2x_3=0.$ $\qquad$ $x_3^2+2x_1x_2=0.$

These which after the transformations of §8 (15) become the real conics have equations,—

In trilinear coörds.	In conjugate coördinates.
$x_1^2+x_2^2+x_3^2=0.$	$2(x^2+xy+y^2)+3=0.$
$x_1^2+\omega x_2^2+\omega^2x_3^2=0.$	$\omega^2x^2-2xy+\omega y^2-2(1-\omega)x-2(1-\omega^2)y=0.$
$x_1^2+\omega^2x_2^2+\omega x_3^2=0.$	$\omega x^2-2xy+\omega^2y^2-2(1-\omega^2)x-2(1-\omega)y=0.$
$x_1^2+2x_2x_3=0.$	$x^2+4xy+y^2+3=0.$

Respectively these have equations as follows

In Cartesian coördinates

Rectangular: with axis of reals as X-axis.	Oblique: the lines through ω and ω² resp. as X and Y axis.
$6X^2-2Y^2+3=0.$	$XY=-\frac{2}{3}.$
$(\sqrt{3}X+Y)^2+2\sqrt{3}(\sqrt{3}X-Y)=0.$	$X^2+2\sqrt{3}Y=0.$
$(\sqrt{3}X-Y)^2+2\sqrt{3}(\sqrt{3}X+Y)=0.$	$Y^2+2\sqrt{3}X=0.$
$6X^2+2Y^2+3=0.$	Nullipartite.

Thus they are readily seen to be respectively

an hyperbola with imaginary lines thru origin as asymptotes.

a parabola passing thru origin and symmetrical to line $\overline{O\omega}$.

" " " " " " " " " $\overline{O\omega^2}$.

25. After the publication in the J. H. U. Circular referred to in §21, the following papers by S. Kantor were found.

In 1895 he notes (¹) the 36 collineations in connection with two triangles in 6-fold perspective in the following, —

"Ueber der Figur zweier 6-fach perspectiver Dreiecke existirt eine Gruppe von 36 ebenen Collineationen. Dies sind die Collineationen, welche ein gewisses [illegible]

(¹) Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene (Berlin) S. 55.

[illegible] C_3 des syzygetischen Büschels z. produciren, das durch den Schnitt der zwei Tripel bestimmt ist. Die Collineationen sind: 9 Inductionen, 8 von Index 3 und die Identität, welche eine invariante Untergruppe zusammensetzen und 18 von index 4."

The following year, Kantor speaks (1) of "jener 36 Kegelschnitte, nach welchen die Wendepunktsconfiguration (das Quadrupel der vier Hesse'schen Dreiecke) sich selbst polar ist," and procedes to define the conics: "Diese 36 Kegelschnitte sind wie folgt vertheilt. Jedes Wendedreieck wird in je zwei Eckpunkten von drei Kegelschnitten, also insgesammt von 9 berührt, von welchen jeder einem zweiten Wendedreiecke conjugirt ist und die beiden jedesmal übrigen zu einander polaren Dreiecken besitzt. Je drei Kegelschnitte also, welche ein Wendedreieck in drei verschiedenen Eckpunktpaaren berühren, haben ein gemeinsames Polardreieck und ein und dasselbe Paar polar conjugirter Dreiecke. Jedoch sind die Kreisdreiecken

(1) Crelle Journal, Bd. 116. ss 171-77

Zuweisungen der beiden letzteren Dreiecke verschieden für die drei Kegelschnitte. Die Correlationsgruppe muss die vier Fundamentaldreiecke untereinander permutiren und kann daher keine anderen Polaritäten als jene 36 enthalten."

These are our 36 conics defined in a different way and used as stated in Rauter's theorem XIII. to operate on the collineations of types 5, 6 and 7 according to Jordan[1] to produce the most general group of correlations which contain the 5th, 6th, and 7th types of Jordan.

By finding the "intermediate" or Salmon's contravariant conic Φ (1) of each two conics in the 36 and remembering what this intermediate is geometrically, we learn the relative positions of the 36 conics. Also this process gives us other sets of line-conics mostly of 27 each but none are the 27 having 6-fold contact with a cubic of the pencil (See Höhere Eb. Kurven § 156, p. 173) as I had hoped. These intermediates are not sufficiently pertinent to give them here.

(1) Crelle Journal, Bd. 84, p. 92. Salmon-Fiedler: Kegs., 6° Aufl. II., S. 668.

Chapter III.

The Hesse Group of 216 Collineations.

26. History and Bibliography.

The attempts to determine all finite groups of transformations run back to the work of F. Klein in the number for July 1874 of the Sitzungsberichte der Erlanger physikalische-medicinische Gesellschaft under the title, "Ueber binäre Formen mit linearen Transformationen in sich selbst." Again in 1876, Math. Ann. 9: s. 185, he sets the Problem, "Alle Gruppen anzugeben, welche aus einer endlichen Anzahl von linearen Transformationen bestehen," and proceeds to determine them by the methods of "Rotationen" of "die regulären Körper — mit sich selbst zur Deckung zu bringen."

In 1876, L. Fuchs in Crelle's Journal 81: 97–142 writes on "Die linearen Differentialgleichungen zweiter Ordnung, welche algebraische Integrale besitzen, und

eine neue Anwendung der Invariantentheorie." and the following year P. Gordan, Math. Ann. 12:23-46, has a paper entitled, "Ueber endliche Gruppen linear Transformationen einer Veränderlichen". These two, near by entirely different methods confirm the results of Klein for the binary domain.

Further, on the groups given by Klein and Jordan, I would call attention to two recent dissertations from the University of Strassburg; viz.

Henner: Ueber die Gruppe der 24 Collineationen durch die ein ebenes Viereck oder ein Vierkant in sich selbst übergeht. — Bayreuth, 1896.

Hugo Himpel: Ueber die Gruppe der 120 Collineationen, durch die ein räumliches Fünfeck in sich selbst übergeht. — Strassburg, 1903.

The groups of both the binary and the ternary domain were tried to be completely given in 1878 by Camille Jordan in a notable memoir,(1) "Mémoire sur les équations différentielles lini-

(1) Crelle Journal. 84:89-215.

aires à integrale algébrique." He is the discoverer of the group G_{216}, mentioned on p. 20 and called the Hessian group. He had noted its existence before the writing here cited as he therein states. Two years later (1880), Jordan devotes himself directly to the group problem in a memoir, in the Atti della Reale Accademia della Scienze Fisiche e Mathematische. Società Reale di Napoli. Vol. VIII. No. 11, pp. 1-41, entitled "Sur la Détermination des Groupes d'Ordre Fini Contenus dans le Groupe Linéaire." It is worthy of note that Jordan by his method did not find the simple group G_{168}, discovered by Klein (Math. Ann. (1879) 14: 428-471).

Next, in 1887, the Hessian group is considered by Alexander Witting in his dissertation: "Ueber eine der Hesse'schen Configuration im Raume, auf welche die Transformationstheorie der hyperelliptischen Functionen (p=2) führt." — Inaug. Diss. Göttingen. 58 S. 8°.

Heinrich Maschke gives the fullest treatment of the subject. See Math. Ann. 29: 157ff.; Nachrichten d. K. Gesellschaft d. Wiss. zu Göttingen, (1888) #5, ss. 78ff.; and especially in connection with "Aufstellung des vollen Formensystems einer quaternären Gruppen von 51840 linearen Substitutionen," Math. Ann. 33: 317–344, (1889).

Again in 1889 we have an important memoir, that by H. Valentiner: "De endelige Transformations-Gruppers Theori," in Copenhagen K. Danske Videnskab. Selskab. Naturvidenskab. og Math. 6te Raekke V: 2, pp. 67–235. French résumé pp. 205–235. Valentiner shows rather clearly in this memoir that he did not know of the one by Jordan and on pp. 151, 222, denies the existence of the group of 216 collineations. He demonstrates (p. 69) the possibility of a group of 72 transformations containing ones of 2nd, 3rd and 4th order, where the transformations of the 4th order by sixes have a common second

doubt. Later he shows the group really exists. He discovered the group G_{360} and presents it, pp 191–195, and 231–233 in French.

A. Wiman refers to the Hessian Group in connection with his paper in Math. Ann. 47:531–556: "Ueber eine einfache Gruppe von 360 ebenen Collineationen" (1896), and, as is to be expected, in 1900 in Encyclopädie der Mathematischen Wissenschaften, Bd. I., H. 5, 3 f (See p. 528), where he writes on "Endliche Gruppen Linearer Substitutionen".

S. Kantor's paper referred to in § 25, though written in 1896, makes no provision for, nor mention of G_{360} discovered at least seven years before.

In the Kansas University Quarterly of January 1901, H. B. Newson presents "The Group of 216 Collineations in the Plane", from the one to eighteen correspondence with the tetrahedral group. Any mention of this group –

-dence and the consequent sub-groups in the ?H.U. Circular was published before the paper by Newson was found, as is shown amongst other statements by the remark that I had not found the G_{54}'s mentioned. Before the publication of my article I saw that Jordan (Atti d. R. Acc. di Napoli, loc. cit. p. 18) said that a group G of order 24·9 4 contains a group H of order 27, and that the substitutions of G are permutable with H. This, G_{27}, sub-group I had not up to the time of publication been able to find. The paper by Newson mentions the G_{27}.

Newson expresses his argument thus: "a cubic C is one of a set of twelve cubics which can be projectively transformed into one another. Since each cubic of the set may be transformed into itself in eighteen different ways (Clebsch: Vorles. ü. Geometrie I. s. 512), we infer that each cubic of the set may also be transformed into any other cubic of the set in eighteen different ways. If this be

type, there are 12·18 transformations which leave invariant the set of twelve cubics. These 216 collineations form a group G_{216}." In relation to the invariant subgroup he says, "The fact that these eighteen transformations form a group may be verified by applying the test of forming all possible resultants."

A better reference than above given by Newson would have been to F. Klein (¹) who in 1871 presents as a new theorem the following:

"Eine allgemeine ebene Curve 3ter Ordnung und insbesondere ihre Wendepunkte durch 18 lineare Transformationen in sich übergehen.

"Durch diese Transformationen geht nun nicht nur die gegebene Curve f, sondern auch ihre Hesse'sche Determinante Δ, überhaupt jede Curve des Büschels $f+\lambda\Delta$ in sich über."

As the quotation shows, Newson's argument is rather inferential, without proof and based on testing

(¹) Math. Ann. 4:354. Sec. III. Die Gleichungen für die Wendepunkte der Curven dritter Ordnung. Die Kreisteilung.

27. Relative to the Hesse Group G_{216}, the purpose is to derive its collineations by purely geometrical processes from certain conics.

Three point-conics such that each is apolar to the remaining two in line-form, namely the parameter be identical in all, give rise to and determine, for all values of this "parameter," the syzygetic pencil of cubics in the Hesse "canonical form."

The three conics

$$x_1^2 + 2m\,x_2 x_3 = 0,$$

$$x_2^2 + 2m\,x_3 x_1 = 0, \qquad (34)$$

$$x_3^2 + 2m\,x_1 x_2 = 0,$$

are of this sort; and it has been shown that in the set of conics determined by three arbitrary conics there are just four sets of three each of this type, and that those of each set are tangent at the vertices two and two to the sides of one of the four inflexional triangles of the cubic of the set. See quotation from Haskell §25.

The Jacobian (¹) of three forms (311) is the cubic curve, the locus of points whose polars as to the three conics meet in a point, which is also on the cubic and has the former point as its corresponding point in the same way as it is of that point. The cubic f is then

$$f \equiv \begin{vmatrix} x_1 & mx_3 & mx_2 \\ mx_3 & x_2 & mx_1 \\ mx_2 & mx_1 & x_3 \end{vmatrix} = 0,$$

or $f \equiv -m^2(x_1^3+x_2^3+x_3^3) + (1+2m^3)x_1x_2x_3 = 0,$ (3)

which is the Hessian, (3) §3, of the cubic (2) whose parameter is $6m$, and so is a member of the syzygetic pencil (as shown in §3) given by equation (2) by giving **m** all positive and negative values from 0 to ∞. Therefore by varying the parameter in equations (314) we have three similar pencils of conics for the vertices two and two which are related to the syzygetic pencil of cubics as above shown.

28. The 36 Conics.

The inflexional triangles of this pencil are

(¹) See C.-L.: I., p. 378.

given by equations (5) and (6) p. 10, and the four equianharmonic cubics have equations (13), p. 14, to define them.

The thirty-six conics are the first polars of the vertices of the inflexional triangles as to the four equianharmonic cubics (12), less twelve degenerate conics which are the squares of the several sides of the inflexional triangles themselves. These twelve come from taking polars as indicated by $A_iS_1, B_iS_2, C_iS_3, D_iS_4$, $(i=1,2,3)$. The subscript *i* indicates the vertex in order found in equations (6), the letters are the respective triangles and equianharmonic cubics.

The conics in this way are as follows:

B_1S_1: $x_1^2+x_2^2+x_3^2=0$. C_1S_1: $\omega x_1^2+x_2^2+x_3^2=0$. D_1S_1: $\omega^2 x_1^2+x_2^2+x_3^2=0$.

B_2S_1: $x_1^2+\omega x_2^2+\omega^2 x_3^2=0$. C_2S_1: $x_1^2+\omega x_2^2+x_3^2=0$. D_2S_1: $x_1^2+\omega^2 x_2^2+x_3^2=0$.

B_3S_1: $x_1^2+\omega^2 x_2^2+\omega x_3^2=0$. C_3S_1: $x_1^2+x_2^2+\omega x_3^2=0$. D_3S_1: $x_1^2+x_2^2+\omega^2 x_3^2=0$.

$$C_1S_2:\ \omega x_1^2+x_2^2+x_3^2+2(\omega x_2x_3+x_3x_1+x_1x_2)=0.$$

$$C_2S_2:\ x_1^2+\omega x_2^2+x_3^2+2(x_2x_3+\omega x_3x_1+x_1x_2)=0.$$

$$C_3S_2:\ x_1^2+x_2^2+\omega x_3^2+2(x_2x_3+x_3x_1+\omega x_1x_2)=0.$$

$$D_1S_2:\quad \omega^2x_1^2+x_2^2+x_3^2+2(\omega^2x_2x_3+x_3x_1+x_1x_2)=0.$$

$$D_2S_2:\quad x_1^2+\omega^2x_2^2+x_3^2+2(x_2x_3+\omega^2x_3x_1+x_1x_2)=0.$$

$$D_3S_2:\quad x_1^2+x_2^2+\omega^2x_3^2+2(x_2x_3+x_3x_1+\omega^2x_1x_2)=0.$$

$$A_1S_2:\quad x_1^2+2x_2x_3=0.$$

$$A_2S_2:\quad x_2^2+2x_3x_1=0.$$

$$A_3S_2:\quad x_3^2+2x_1x_2=0.$$

$$D_1S_3:\quad \omega^2x_1^2+x_2^2+x_3^2+2\omega^2(\omega^2x_2x_3+x_3x_1+x_1x_2)=0.$$

$$D_2S_3:\quad x_1^2+\omega^2x_2^2+x_3^2+2\omega^2(x_2x_3+\omega^2x_3x_1+x_1x_2)=0.$$

$$D_3S_3:\quad x_1^2+x_2^2+\omega^2x_3^2+2\omega^2(x_2x_3+x_3x_1+\omega^2x_1x_2)=0.$$

$$A_1S_3:\quad x_1^2+2\omega^2x_2x_3=0.$$

$$A_2S_3:\quad x_2^2+2\omega^2x_3x_1=0.$$

$$A_3S_3:\quad x_3^2+2\omega^2x_1x_2=0.$$

$$B_1S_3:\quad x_1^2+x_2^2+x_3^2+2\omega^2(x_2x_3+x_3x_1+x_1x_2)=0.$$

$$B_2S_3:\quad x_1^2+\omega x_2^2+\omega^2x_3^2+2\omega^2(x_2x_3+\omega x_3x_1+\omega^2x_1x_2)=0.$$

$$B_3S_3:\quad x_1^2+\omega^2x_2^2+\omega x_3^2+2\omega^2(x_2x_3+\omega^2x_3x_1+\omega x_1x_2)=0.$$

$$A_1S_4:\quad x_1^2+2\omega x_2x_3=0.$$

$$A_2S_4:\quad x_2^2+2\omega x_3x_1=0.$$

$$A_3S_4:\quad x_3^2+2\omega x_1x_2=0.$$

$$B_1S_4:\quad x_1^2+x_2^2+x_3^2+2\omega(x_2x_3+x_3x_1+x_1x_2)=0.$$

$$B_2S_4:\quad x_1^2+\omega x_2^2+\omega^2x_3^2+2\omega(x_2x_3+\omega x_3x_1+\omega^2x_1x_2)=0.$$

$$B_3S_4:\quad x_1^2+\omega^2x_2^2+\omega x_3^2+2\omega(x_2x_3+\omega^2x_3x_1+\omega x_1x_2)=0.$$

$$C_1S_4:\quad \omega x_1^2+x_2^2+x_3^2+2\omega(\omega x_2x_3+x_3x_1+x_1x_2)=0.$$

$$C_2S_4:\quad x_1^2+\omega x_2^2+x_3^2+2\omega(x_2x_3+\omega x_3x_1+x_1x_2)=0.$$

$$C_3S_4:\quad x_1^2+x_2^2+\omega x_3^2+2\omega(x_2x_3+x_3x_1+\omega x_1x_2)=0.$$

These conics form a closed system as shown in §22, p. 35; and are in the same order (except B_2S_3 and B_3S_3 [illegible]) as derived in the way explained in §21. Further, the line-forms of these are given by the same matrices as the point-forms as a whole. Any conic's matrix for its line-form is that of its point-form with ω and ω^2 exchanged. The matrices of point- and of line-forms which are alike are,

$B_iS_1\equiv\mathcal{B}_{1,3,2}\mathcal{S}_1,\quad C_iS_1\equiv\mathcal{D}_i\mathcal{S}_1,\quad D_iS_1\equiv\mathcal{C}_i\mathcal{S}_1,$

$C_iS_2\equiv\mathcal{D}_i\mathcal{S}_2,\quad D_iS_2\equiv\mathcal{C}_i\mathcal{S}_2,\quad A_iS_2\equiv\mathcal{A}_i\mathcal{S}_2,$

$D_iS_3\equiv\mathcal{C}_i\mathcal{S}_4,\quad A_iS_3\equiv\mathcal{A}_i\mathcal{S}_4,\quad B_iS_3\equiv\mathcal{B}_{1,3,2}\mathcal{S}_4,$

$A_iS_4\equiv\mathcal{A}_i\mathcal{S}_3,\quad B_iS_4\equiv\mathcal{B}_{1,3,2}\mathcal{S}_3,\quad C_iS_4\equiv\mathcal{D}_i\mathcal{S}_3,$

where $i=1,2,3$, Roman caps stand for point-conics and

script letters stand for line-conics.

29. These Conics as Source of the 216 Collineations.

The product of a point-conic on a line-conic is a collineation. With this in mind we form a multiplication-table with the point-conics along the left side and the line-conics along the top. The product of any point-conic on any line-conic is a collineation, which was put in matrix form into the square of the corresponding row and column. Thus arranged, the collineations were readily classified, from their actions on the inflexional triangles, into the tetrahedral subgroups. In the following table we put the numbers of the collineations corresponding to the numbers above the collineations enumerated after the table.

In the multiplication of matrices, remember that the separate terms of the rows upon those of the columns of multiplicand give rows of product.

FOLD OUT

30. The 216 Collineations.

1. $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
2. $\begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix}$
3. $\begin{pmatrix}1&0&0\\0&\omega^2&0\\0&0&\omega\end{pmatrix}$
4. $\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$
5. $\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$
6. $\begin{pmatrix}0&0&1\\\omega^2&0&0\\0&\omega&0\end{pmatrix}$
7. $\begin{pmatrix}0&1&0\\0&0&\omega\\\omega^2&0&0\end{pmatrix}$
8. $\begin{pmatrix}0&1&0\\0&0&\omega^2\\\omega&0&0\end{pmatrix}$
9. $\begin{pmatrix}0&0&1\\\omega&0&0\\0&\omega^2&0\end{pmatrix}$
10. $\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$
11. $\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$
12. $\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$ G_{18}.
13. $\begin{pmatrix}1&0&0\\0&0&\omega\\0&\omega^2&0\end{pmatrix}$
14. $\begin{pmatrix}0&0&1\\0&\omega&0\\\omega^2&0&0\end{pmatrix}$
15. $\begin{pmatrix}0&1&0\\\omega&0&0\\0&0&\omega^2\end{pmatrix}$
16. $\begin{pmatrix}1&0&0\\0&0&\omega^2\\0&\omega&0\end{pmatrix}$
17. $\begin{pmatrix}0&0&1\\0&\omega^2&0\\\omega&0&0\end{pmatrix}$
18. $\begin{pmatrix}0&1&0\\\omega^2&0&0\\0&0&\omega\end{pmatrix}$

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19. $\begin{pmatrix}1&1&1\\1&\omega^2&\omega\\1&\omega&\omega^2\end{pmatrix}$
20. $\begin{pmatrix}\omega^2&1&\omega\\1&1&1\\\omega&1&\omega^2\end{pmatrix}$
21. $\begin{pmatrix}\omega^2&\omega&1\\\omega&\omega^2&1\\1&1&1\end{pmatrix}$
22. $\begin{pmatrix}1&1&1\\\omega&1&\omega^2\\\omega^2&1&\omega\end{pmatrix}$
23. $\begin{pmatrix}\omega&\omega^2&1\\1&1&1\\\omega^2&\omega&1\end{pmatrix}$
24. $\begin{pmatrix}1&\omega^2&\omega\\1&\omega&\omega^2\\1&1&1\end{pmatrix}$
25. $\begin{pmatrix}1&1&1\\\omega^2&\omega&1\\\omega&\omega^2&1\end{pmatrix}$
26. $\begin{pmatrix}1&\omega&\omega^2\\1&1&1\\1&\omega^2&\omega\end{pmatrix}$
27. $\begin{pmatrix}\omega&1&\omega^2\\\omega^2&1&\omega\\1&1&1\end{pmatrix}$
28. $\begin{pmatrix}1&1&1\\1&\omega&\omega^2\\1&\omega^2&\omega\end{pmatrix}$
29. $\begin{pmatrix}\omega&1&\omega^2\\1&1&1\\\omega^2&1&\omega\end{pmatrix}$
30. $\begin{pmatrix}\omega&\omega^2&1\\\omega^2&\omega&1\\1&1&1\end{pmatrix}$
31. $\begin{pmatrix}1&1&1\\\omega&\omega^2&1\\\omega^2&\omega&1\end{pmatrix}$
32. $\begin{pmatrix}1&\omega^2&\omega\\1&1&1\\1&\omega&\omega^2\end{pmatrix}$
33. $\begin{pmatrix}\omega^2&1&\omega\\\omega&1&\omega^2\\1&1&1\end{pmatrix}$
34. $\begin{pmatrix}1&1&1\\\omega^2&1&\omega\\\omega&1&\omega^2\end{pmatrix}$
35. $\begin{pmatrix}\omega^2&\omega&1\\1&1&1\\\omega&\omega^2&1\end{pmatrix}$
36. $\begin{pmatrix}1&\omega&\omega^2\\1&\omega^2&\omega\\1&1&1\end{pmatrix}$

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37. $\begin{pmatrix}\omega^2&1&1\\\omega&1&\omega\\\omega&\omega&1\end{pmatrix}$
38. $\begin{pmatrix}\omega&\omega&1\\\omega^2&1&1\\\omega&1&\omega\end{pmatrix}$
39. $\begin{pmatrix}\omega&1&\omega\\\omega&\omega&1\\\omega^2&1&1\end{pmatrix}$
40. $\begin{pmatrix}1&\omega&\omega\\1&\omega^2&1\\\omega&\omega&1\end{pmatrix}$
41. $\begin{pmatrix}\omega&\omega&1\\1&\omega&\omega\\1&\omega^2&1\end{pmatrix}$
42. $\begin{pmatrix}1&\omega^2&1\\\omega&\omega&1\\1&\omega&\omega\end{pmatrix}$
43. $\begin{pmatrix}1&\omega&\omega\\\omega&1&\omega\\1&1&\omega^2\end{pmatrix}$
44. $\begin{pmatrix}1&1&\omega^2\\1&\omega&\omega\\\omega&1&\omega\end{pmatrix}$
45. $\begin{pmatrix}\omega&1&\omega\\1&1&\omega^2\\1&\omega&\omega\end{pmatrix}$
46. $\begin{pmatrix}\omega^2&1&1\\\omega&\omega&1\\\omega&1&\omega\end{pmatrix}$
47. $\begin{pmatrix}1&1&\omega^2\\\omega&1&\omega\\1&\omega&\omega\end{pmatrix}$
48. $\begin{pmatrix}1&\omega^2&1\\1&\omega&\omega\\\omega&\omega&1\end{pmatrix}$
49. $\begin{pmatrix}1&\omega&\omega\\1&1&\omega^2\\\omega&1&\omega\end{pmatrix}$
50. $\begin{pmatrix}\omega&\omega&1\\1&\omega^2&1\\1&\omega&\omega\end{pmatrix}$
51. $\begin{pmatrix}1&\omega^2&1\\\omega&\omega^2&\omega^2\\1&1&\omega^2\end{pmatrix}$
52. $\begin{pmatrix}\omega^2&1&1\\1&1&\omega^2\\\omega^2&\omega&\omega^2\end{pmatrix}$
53. $\begin{pmatrix}1&1&\omega^2\\1&\omega^2&1\\\omega&\omega^2&\omega^2\end{pmatrix}$
54. $\begin{pmatrix}1&\omega^2&1\\\omega^2&1&1\\\omega^2&\omega^2&\omega\end{pmatrix}$

55. $\begin{pmatrix} \omega & 1 & 1 \\ \omega^2 & 1 & \omega^2 \\ \omega^2 & \omega^2 & 1 \end{pmatrix}$
56. $\begin{pmatrix} \omega^2 & \omega^2 & 1 \\ \omega & 1 & 1 \\ \omega^2 & 1 & \omega^2 \end{pmatrix}$
57. $\begin{pmatrix} \omega^2 & 1 & \omega^2 \\ \omega^2 & \omega^2 & 1 \\ \omega & 1 & 1 \end{pmatrix}$
58. $\begin{pmatrix} 1 & \omega^2 & \omega^2 \\ 1 & \omega & 1 \\ \omega^2 & \omega^2 & 1 \end{pmatrix}$
59. $\begin{pmatrix} \omega^2 & \omega^2 & 1 \\ 1 & \omega^2 & \omega^2 \\ 1 & \omega & 1 \end{pmatrix}$
60. $\begin{pmatrix} 1 & \omega & 1 \\ \omega^2 & \omega^2 & 1 \\ 1 & \omega^2 & \omega^2 \end{pmatrix}$

61. $\begin{pmatrix} \omega & 1 & 1 \\ 1 & \omega & 1 \\ \omega & \omega & \omega^2 \end{pmatrix}$
62. $\begin{pmatrix} 1 & 1 & \omega \\ 1 & \omega^2 & \omega^2 \\ \omega^2 & 1 & \omega^2 \end{pmatrix}$
63. $\begin{pmatrix} 1 & \omega & 1 \\ \omega & \omega & \omega^2 \\ \omega & 1 & 1 \end{pmatrix}$
64. $\begin{pmatrix} \omega & 1 & 1 \\ \omega^2 & \omega^2 & 1 \\ \omega^2 & 1 & \omega^2 \end{pmatrix}$
65. $\begin{pmatrix} 1 & 1 & \omega \\ \omega^2 & 1 & \omega^2 \\ 1 & \omega^2 & \omega^2 \end{pmatrix}$
66. $\begin{pmatrix} 1 & \omega & 1 \\ 1 & \omega^2 & \omega^2 \\ \omega^2 & \omega^2 & 1 \end{pmatrix}$

67. $\begin{pmatrix} 1 & \omega^2 & \omega^2 \\ 1 & 1 & \omega \\ \omega^2 & 1 & \omega^2 \end{pmatrix}$
68. $\begin{pmatrix} 1 & 1 & \omega \\ \omega & \omega^2 & \omega \\ \omega & 1 & 1 \end{pmatrix}$
69. $\begin{pmatrix} 1 & \omega & 1 \\ \omega^2 & \omega & \omega \\ 1 & 1 & \omega \end{pmatrix}$
70. $\begin{pmatrix} \omega & 1 & 1 \\ 1 & 1 & \omega \\ \omega & \omega^2 & \omega \end{pmatrix}$
71. $\begin{pmatrix} 1 & 1 & \omega \\ 1 & \omega & 1 \\ \omega^2 & \omega & \omega \end{pmatrix}$
72. $\begin{pmatrix} 1 & \omega & 1 \\ \omega & 1 & 1 \\ \omega & \omega & \omega^2 \end{pmatrix}$

73. $\begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
74. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & \omega^2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
75. $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$
76. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega^2 \\ 0 & 1 & 0 \end{pmatrix}$
77. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}$
78. $\begin{pmatrix} 0 & \omega^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

79. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \omega^2 & 0 \end{pmatrix}$
80. $\begin{pmatrix} 0 & 0 & \omega^2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
81. $\begin{pmatrix} 0 & 1 & 0 \\ \omega^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
82. $\begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
83. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
84. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$

85. $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix}$
86. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix}$
87. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \omega^2 & 0 & 0 \end{pmatrix}$
88. $\begin{pmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
89. $\begin{pmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
90. $\begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

91. $\begin{pmatrix} \omega & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
92. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & \omega & 0 \\ 1 & 0 & 0 \end{pmatrix}$
93. $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \omega \end{pmatrix}$
94. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \omega & 0 \end{pmatrix}$
95. $\begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
96. $\begin{pmatrix} 0 & 1 & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

97. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & 1 & 0 \end{pmatrix}$
98. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \omega & 0 & 0 \end{pmatrix}$
99. $\begin{pmatrix} 0 & \omega & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
100. $\begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
101. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$
102. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}$

103. $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}$
104. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix}$
105. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \omega & 0 & 0 \end{pmatrix}$
106. $\begin{pmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
107. $\begin{pmatrix} 0 & 0 & 1 \\ \omega & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
108. $\begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

109. $$\begin{pmatrix} \omega^2 & 1 & 1 \\ 1 & 1 & \omega^2 \\ 1 & \omega^2 & 1 \end{pmatrix}$$
110. $$\begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega^2 & 1 \\ \omega^2 & 1 & 1 \end{pmatrix}$$
111. $$\begin{pmatrix} 1 & \omega^2 & 1 \\ \omega^2 & 1 & 1 \\ 1 & 1 & \omega^2 \end{pmatrix}$$
112. $$\begin{pmatrix} 1 & \omega & \omega \\ \omega^2 & \omega^2 & \omega \\ 1 & \omega^2 & 1 \end{pmatrix}$$
113. $$\begin{pmatrix} 1 & 1 & \omega^2 \\ \omega & 1 & \omega \\ \omega & \omega^2 & \omega^2 \end{pmatrix}$$
114. $$\begin{pmatrix} 1 & \omega^2 & 1 \\ 1 & \omega & \omega \\ \omega^2 & \omega^2 & \omega \end{pmatrix}$$
115. $$\begin{pmatrix} 1 & \omega & \omega \\ 1 & 1 & \omega^2 \\ \omega^2 & \omega & \omega^2 \end{pmatrix}$$
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163. $\begin{pmatrix} 1 & \omega^2 & \omega^2 \\ 1 & \omega & 1 \\ 1 & 1 & \omega \end{pmatrix}$ 164. $\begin{pmatrix} 1 & 1 & \omega \\ \omega & 1 & 1 \\ \omega^2 & 1 & \omega^2 \end{pmatrix}$ 165. $\begin{pmatrix} 1 & \omega & 1 \\ \omega^2 & \omega^2 & 1 \\ \omega & 1 & 1 \end{pmatrix}$ 166. $\begin{pmatrix} \omega & 1 & 1 \\ \omega^2 & 1 & \omega^2 \\ 1 & 1 & \omega \end{pmatrix}$ 167. $\begin{pmatrix} \omega^2 & \omega^2 & 1 \\ \omega & 1 & 1 \\ 1 & \omega & 1 \end{pmatrix}$ 168. $\begin{pmatrix} 1 & \omega & 1 \\ 1 & 1 & \omega \\ 1 & \omega^2 & \omega^2 \end{pmatrix}$

169. $\begin{pmatrix} \omega & 1 & 1 \\ 1 & \omega & 1 \\ \omega^2 & \omega^2 & 1 \end{pmatrix}$ 170. $\begin{pmatrix} 1 & 1 & \omega \\ 1 & \omega^2 & \omega^2 \\ 1 & \omega & 1 \end{pmatrix}$ 171. $\begin{pmatrix} \omega^2 & 1 & \omega^2 \\ 1 & 1 & \omega \\ \omega & 1 & 1 \end{pmatrix}$ 172. $\begin{pmatrix} 1 & \omega^2 & \omega^2 \\ 1 & 1 & \omega \\ 1 & \omega & 1 \end{pmatrix}$ 173. $\begin{pmatrix} \omega^2 & \omega^2 & 1 \\ 1 & \omega & 1 \\ \omega & 1 & 1 \end{pmatrix}$ 174. $\begin{pmatrix} \omega^2 & 1 & \omega^2 \\ \omega & 1 & 1 \\ 1 & 1 & \omega \end{pmatrix}$

175. $\begin{pmatrix} \omega & 1 & 1 \\ 1 & \omega & 1 \\ \omega^2 & \omega^2 & 1 \end{pmatrix}$ 176. $\begin{pmatrix} 1 & 1 & \omega \\ \omega^2 & 1 & \omega^2 \\ \omega & 1 & 1 \end{pmatrix}$ 177. $\begin{pmatrix} 1 & \omega & 1 \\ 1 & \omega^2 & \omega^2 \\ 1 & 1 & \omega \end{pmatrix}$ 178. $\begin{pmatrix} \omega & 1 & 1 \\ 1 & 1 & \omega \\ \omega^2 & 1 & \omega^2 \end{pmatrix}$ 179. $\begin{pmatrix} 1 & 1 & \omega \\ 1 & \omega & 1 \\ 1 & \omega^2 & \omega^2 \end{pmatrix}$ 180. $\begin{pmatrix} 1 & \omega & 1 \\ \omega & 1 & 1 \\ \omega^2 & \omega^2 & 1 \end{pmatrix}$

181. $\begin{pmatrix} 1 & 1 & 1 \\ \omega^2 & 1 & \omega \\ \omega^2 & \omega & 1 \end{pmatrix}$ 182. $\begin{pmatrix} \omega^2 & \omega & 1 \\ 1 & 1 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix}$ 183. $\begin{pmatrix} \omega^2 & 1 & \omega \\ \omega^2 & \omega & 1 \\ 1 & 1 & 1 \end{pmatrix}$ 184. $\begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ 1 & \omega^2 & \omega \end{pmatrix}$ 185. $\begin{pmatrix} 1 & \omega^2 & \omega \\ 1 & 1 & 1 \\ \omega & \omega^2 & 1 \end{pmatrix}$ 186. $\begin{pmatrix} \omega & \omega^2 & 1 \\ 1 & \omega^2 & \omega \\ 1 & 1 & 1 \end{pmatrix}$

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193. $\begin{pmatrix} 1 & 1 & 1 \\ \omega & 1 & \omega^2 \\ 1 & \omega & \omega^2 \end{pmatrix}$ 194. $\begin{pmatrix} \omega & \omega^2 & 1 \\ 1 & 1 & 1 \\ 1 & \omega^2 & \omega \end{pmatrix}$ 195. $\begin{pmatrix} \omega^2 & \omega & 1 \\ \omega^2 & 1 & \omega \\ 1 & 1 & 1 \end{pmatrix}$ 196. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ \omega & \omega^2 & 1 \end{pmatrix}$ 197. $\begin{pmatrix} \omega^2 & 1 & \omega \\ 1 & 1 & 1 \\ \omega^2 & \omega & 1 \end{pmatrix}$ 198. $\begin{pmatrix} \omega & 1 & \omega^2 \\ 1 & \omega & \omega^2 \\ 1 & 1 & 1 \end{pmatrix}$

199. $\begin{pmatrix} 1 & \omega & \omega \\ 1 & 1 & \omega^2 \\ 1 & \omega^2 & 1 \end{pmatrix}$ 200. $\begin{pmatrix} \omega & \omega & 1 \\ 1 & \omega^2 & 1 \\ \omega^2 & 1 & 1 \end{pmatrix}$ 201. $\begin{pmatrix} \omega & 1 & \omega \\ \omega^2 & 1 & 1 \\ 1 & 1 & \omega^2 \end{pmatrix}$ 202. $\begin{pmatrix} \omega^2 & 1 & 1 \\ \omega & \omega & 1 \\ 1 & \omega^2 & 1 \end{pmatrix}$ 203. $\begin{pmatrix} 1 & 1 & \omega^2 \\ \omega & 1 & \omega \\ \omega^2 & 1 & 1 \end{pmatrix}$ 204. $\begin{pmatrix} 1 & \omega^2 & 1 \\ 1 & \omega & \omega \\ 1 & 1 & \omega^2 \end{pmatrix}$

205. $\begin{pmatrix} \omega^2 & 1 & 1 \\ 1 & 1 & \omega^2 \\ \omega & 1 & \omega \end{pmatrix}$ 206. $\begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega^2 & 1 \\ 1 & \omega & \omega \end{pmatrix}$ 207. $\begin{pmatrix} 1 & \omega^2 & 1 \\ \omega^2 & 1 & 1 \\ \omega & \omega & 1 \end{pmatrix}$ 208. $\begin{pmatrix} 1 & \omega & \omega \\ 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \end{pmatrix}$ 209. $\begin{pmatrix} 1 & 1 & \omega^2 \\ \omega^2 & 1 & 1 \\ \omega & 1 & \omega \end{pmatrix}$ 210. $\begin{pmatrix} 1 & \omega^2 & 1 \\ \omega & \omega & 1 \\ \omega^2 & 1 & 1 \end{pmatrix}$

211. $\begin{pmatrix} \omega^2 & 1 & 1 \\ \omega & 1 & \omega \\ 1 & 1 & \omega^2 \end{pmatrix}$ 212. $\begin{pmatrix} \omega & \omega & 1 \\ \omega^2 & 1 & 1 \\ 1 & \omega^2 & 1 \end{pmatrix}$ 213. $\begin{pmatrix} 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \\ 1 & \omega & \omega \end{pmatrix}$ 214. $\begin{pmatrix} \omega^2 & 1 & 1 \\ 1 & \omega^2 & 1 \\ \omega & \omega & 1 \end{pmatrix}$ 215. $\begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ 1 & \omega^2 & 1 \end{pmatrix}$ 216. $\begin{pmatrix} \omega & 1 & \omega \\ 1 & 1 & \omega^2 \\ \omega^2 & 1 & 1 \end{pmatrix}$

31. From the system of generators of the G_{216} as given by Maschke (Math. Ann. 33:324) we see that all the collineations can be deduced from six of the conics in line-form and two of these six in point-form. These line-conics are A_1S_1, $A_{1,2}S_2$, and C_1S_4; the point-conics are A_1S_1, A_1S_2. All the 216 collineations can be deduced from these conics, and no other collineations arise therefrom.

The Tetrahedral Sub-Group.

32. The sub-groups arising from the one-to-eighteen correspondence with the tetrahedral group in the plane [§26, p. 149] are these;—

to the identical transformation corresponds 1 G_{18},

" " 3 G_2's correspond 3 G_{36}'s,

" " 4 G_3's " 4 G_{54}'s,

" " 1 G_4 corresponds 1 G_{72}.

We naturally transform the four triangles thus,

the triangles A B C D are sent respectively

into ") A B C D by collineations 1–18,

" " B A D C " " 19–36,

" " C D A B " " 37–54,

" " D C B A " " 55–72.

"The first eighteen form the identity G_{18}; the other 3 sets of 18 each, each with the collineations of the G_{18} form one of the three G_{36}'s which interchange the triangles by twos. All these form the one G_{72} which permute all four triangles.

Next, the triangles are permuted as follows;

triangles	A B C D	are sent respectively into			
"	A D B C A C D B	by 36 collineations, numbers			73 – 90, 91 – 108,
	D B A C C B D A	" 36	"	, "	109 – 126, 127 – 144,
	D A C B B D C A	" 36	"	, "	145 – 162, 163 – 180,
	C A B D B C A D	" 36	"	, "	181 – 198, 199 – 216.

The collineations of each of these sets in connection with the G_{18} form a sub-group G_{54}, leaving one triangle unaltered and permuting the other three cyclically.

All the collineations are readily classified by testing with simplest triangle, viz., A.

The transformations of the G_{18} of course send each respective inflexional triangle into

itself, but not point for point in each case. A comparison of the G_{18} with respect to triangle A at once shows that, since A ..., therefore, each triangle is sent by three collineations identically into itself. These are identity and two of the eight collineations of period three. The nine collineations of period 2 divide, differently for each triangle, into sets of three. Those of each set act the same on that triangle, sending one vertex into itself and interchanging the other two; that is, as appears by comparing equations (7), one of the three harmonic polars on the fixed vertex remains a fixed line of the collineation; and the other two harmonic polars are interchanged.

Other Sub-Groups.

33. Notice that the conics naturally, from our definition of them, divide into sets of nine, each with three sub-sets of three each.

By examining the collineations of the G_{18}

we readily see that numbers 1, 4, 5, 10, 11, 12 form a dihedral group, G_6, sending conic B_1S_1 into itself. Hence from symmetry, there are $\frac{36}{3}$ or 12 dihedral G_6's. These each contain the identity collineation, two of period 3 and three of period 2; there is thus in each a cyclic G_3 which is an invariant sub-group. The transform of one of the collineations of period 2 by one of the G_3 is one of those of period 2. ——

The G_{18} contains the identity, eight collineations of period three and 9 of period two. The former compose four cyclic G_3's, the latter 9 cyclic G_2's.

Next, we observe that the collineations of the three sets of 18 each, which with the G_{18} form the G_{72}, are all of period four. These by sixes have a common second power. These nine second powers are the nine of period two in the G_{18}. Thus we have what Valentiner mentions loc. cit, p. 69:

"It is thus demonstrated that there [illegible]

exists a group of 72 transformations, containing forms of 2°, 3°, 4° order, where the transformations of 4° order 6 and 6 have a common second power. This group will be shown, really to exist." To every transformation of second order, the two of order four and identity form a cyclic sub-group of order 4. Thus there are 27 cyclic G_4's.

It seems strange that Valentiner should have overlooked and even practically denied the existence of the G_{216}, when he knew there must be 54 transformations of period four, and states that that number ought to be $\frac{1}{4}N$, N representing the total number of transformations. He further gets the G_{72} with its sub-groups. I am inclined to question the complete accuracy of the French résumé, for at end of § 45, pp. 173, 227, the Danish and French are exactly contradictory.

The cyclic G_4's divide the conics into

sets of two and four, 4 only belonging to each of two different natural sets of three. The group permutes those of one set into those of the other.

In each of the eight sets of 18 each, which remain to be considered, 9 collineations are of period three and 9 of period six. The first nine are of period six, the second nine are of period three. These latter of each lot of 36 as previously enumerated, with identity form 9 cyclic G_3's or in all 36 more cyclic G_3's.

A cyclic G_6, different from the the dihedral G_6, contains six collineations,

2 of period 6, 2 of period 3, one of period 2, and identity. Thus composed we find there are in the whole Hesse group 12 cyclic G_6's. One of the collineations of period 2 appears in each of the 9 G_6's containing the collineations of any and every set of 36 mentioned above, and each of those of period six appears once.

Jordan in his paper previously cited

on p. 18 says that a group G belonging to this type (Hessian, of order 24·9) contains a group H of order 27. The substitutions of G are permutable with H." In each G_{54} there is such a sub-group, consisting of the 18 collineations of period three in each set of 36 under present consideration, the eight of period three and identity from the G_{18}. So further we have 4 G_{27}'s.

Three of the collineations of period three (the symmetrical or two skew symmetrical ones) from each lot of 36 together with two of the eight of order 3 in G_{18}, and identity, form an Abelian G_9. Thus there are four Abelian G_9's.

34. To recapitulate there are in the Hesse group, 9 cyclic G_2's, 40 cyclic G_3's, 27 cyclic G_4's, 36 dihedral G_6's, 4 Abelian G_9's, and 4 G_{27}'s. These all arrange themselves in interesting form on the multiplication table. The classification as to periodicity is 1 identity, 9 of period 2, 80 of period 3, 54 of 4, 72 of 6.

Chapter IV.

Perspective Triangles. — The complete Pappus Hexagon.

35. Having treated so fully the relations of the four inflexional triangles, which are [§7, p.12] syzygetic or in six-fold perspective, we should show how to obtain sets of triangles of the other perspective forms for convenient use in other analytic work. Especially triply perspective triangles and incidentally two-fold and four-fold perspective ones will be considered.

36. Historically, it is of interest to note some papers. In 1870, H. Schröter [Ueber persp. lieg. Dreiecke, Math. Ann. 2, ss. 553–562] set for himself the problem whether a triangle can be in more than single perspective with another, and in what-fold perspective. His paper is characteristically clear and easy of reading; its method is synthetic and presents a construction for triangles in all possible forms

of perspectivity, viz., one-, two-, three-, four-, and six-fold.

J. Vályi whose paper was referred to in § 21, p. 39, set for himself in 1882 the same problem and reaches the same results analytically without reference to Schröter. Some other papers directly or indirectly presenting perspective triangles are here simply noted:

Rosanes: Ueber Dreiecke in persp. Lage, Math. Ann. 2:549.

Hess: Beiträge z. Theorie d. mehrfach persp. Dreiecke, Ib. 28:167.

Third: Triangles triply in Perspective, Proc. Edinb. M.S., XIX (1901) p. 10.

G. Kohn: Desmische Vierseiten- und Kegelschnitt-systeme, Monatshefte XIV. (1903) s. 74. "First part especially interesting."

G. Kohn: Persp. Tetraeder, Archiv d. Math., II., 6, (1887).

M. Pasch: Ueber Vier-eck und -seit, Math. Ann. 26:211-216.

Caporali: Memorie, pp. 236, 252.

Veronese: Sull'Hexagrammum mysticum, Lincei Mem. III., 1, (1877).

Triply Perspective Triangles in Circular Coördinates.

37 It is well known that two concentric equilateral triangles are triply perspective. We take two such first as point-triads, with coördinates of vertices

1	2	3		1′	2′	3′	respectively
1	ω	ω^2		at	$\omega^2 at$	ωat.	

These two point-triads are in perspective thus

	1	2	3	with Center	and Axis.
with	1′	2′	3′	α	A
	2′	3′	1′	β	B
	3′	1′	2′	γ	Γ.

We name vertices of triangle of axes opposite correspondingly named sides $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$. Lay out Fig. 4 opposite p. 77.

The line $\overline{11'}$ has coörds., $1-\frac{a}{t}$, $at-1$, $\frac{a}{t}-at$,

" " $\overline{22'}$ " " $\omega^2-\frac{\omega a}{t}$, $\omega^2 at-\omega$, $\frac{\omega^2 a}{t}-\omega at$,

" " $\overline{33'}$ " " $\omega-\frac{\omega^2 a}{t}$, $\omega at-\omega^2$, $\frac{\omega a}{t}-\omega^2 at$.

The determinant of these coördinates vanishes for the sum of each column is zero; therefore, the lines meet in a point, α. Its coördinate is readily found to be

$$\alpha \equiv x = \frac{a(at^3-1)}{t(1-a^2)}. \qquad (35)$$

Similarly we deduce,

$$\beta \equiv x = \frac{\omega a(at^3-1)}{t(1-a^2)}. \qquad (35)$$

$$\gamma \equiv x = \frac{\omega^2 a(at^3-1)}{t(1-a^2)}.$$

Evidently these three centers are on a circle concentric with the circles about the original triads and ordered as triad 1, 2, 3. The radius of their circle is at once calculated to be

$$\frac{a}{1-a^2}\sqrt{\frac{(at^3-1)(a-t^3)}{t^3}}. \qquad (35')$$

With radii of the given circumcircles resp. a and b the derived circle has radius

$$\frac{ab}{a^2-b^2}\sqrt{\frac{(at^3-b)(a-bt^3)}{t^3}}.$$

38. The coördinates of sides opposite the vertices 1, 2, 3, respectively are proportional to

$$(1,\ 1,\ 1), \qquad (1,\ \omega^2,\ \omega), \qquad (1,\ \omega,\ \omega^2). \qquad (36)$$

Those of sides opposite 1', 2', 3', resp. are

$$(1,\ t^2,\ at), \qquad (1,\ \omega t^2,\ \omega^2 at), \qquad (1,\ \omega^2 t^2,\ \omega at). \qquad (37)$$

Taking these sides in the same order of perspection we deduce the coördinates of the

axis of perspective. First, from coördinates (36), (37), we write off at once

$$x_{11'} = \frac{at - t^2}{t^2 - 1}, \quad x_{22'} = \frac{\omega at - \omega^2 t^2}{\omega t^2 - \omega^2}, \quad x_{33'} = \frac{\omega^2 at - \omega t^2}{\omega^2 t^2 - \omega}.$$

These three points lie on a line for we have

$$\begin{vmatrix} at - t^2 & 1 - at & t^2 - 1 \\ \omega at - \omega^2 t^2 & \omega - \omega^2 at & \omega t^2 - \omega^2 \\ \omega^2 at - \omega t^2 & \omega^2 - \omega at & \omega^2 t^2 - \omega \end{vmatrix} = 0$$

manifestly. Whence we see that the axis A is

$$[at^3 - 1, \quad t(a - t^3), \quad t^2(1 - a^2)].$$

In like manner the coördinates of B and Γ are found to be

$$[at^3 - 1, \quad \omega^2 t(a - t^3), \quad \omega t^2(1 - a^2)], \qquad (38)$$

and

$$[at^3 - 1, \quad \omega t(a - t^3), \quad \omega^2 t^2(1 - a^2)].$$

From these coördinates we get the vertices of the triangle, ABC, thus,

the meet of BΓ is

$$A \equiv x = \frac{\begin{vmatrix} \omega^2 t(a - t^3) & \omega t^2(1 - a^2) \\ \omega t(a - t^3) & \omega^2 t^2(1 - a^2) \end{vmatrix}}{\begin{vmatrix} at^3 - 1 & \omega^2 t(a - t^3) \\ at^3 - 1 & \omega t(a - t^3) \end{vmatrix}} = \frac{(1 - a^2)t^2}{at^3 - 1}.$$

(39)

Similarly we find

$$B \equiv x = \frac{\omega(1 - a^2)t^2}{at^3 - 1}; \quad C \equiv x = \frac{\omega^2(1 - a^2)t^2}{at^3 - 1}.$$

Manifestly these points form an equilateral triangle, concentric with the other three,

and ordered cyclically with 1, 2, 3. The radius of its circumcircle is at once seen to be

$$(1-a^2)\sqrt{\frac{t^3}{(at^3-1)(a-t^3)}} \qquad (40)$$

The radii of the four circles are seen to form the proportion

$$1 : \frac{a}{1-a^2}\sqrt{\frac{(at^3-1)(a-t^3)}{t^3}} = (1-a^2)\sqrt{\frac{t^3}{(at^3-1)(a-t^3)}} : a \; ;$$

or in other words, the derived circles are a pair mutually inverse as to the same circle to which the original circles are inverse, and its radius is the mean proportional of the radii of original circles.

Therefore, we may state the theorem,

<u>Two concentric equilateral triangles are in triple perspective with their 3 centres and their three axes equiangular triads concentric with the original two, and the radii of the circumcircles of the latter two triads are functions of the radii of the original circumcircles and the clinant of the angle between the given triads.</u>

<u>The product of the radii of the latter two circumcircles equals the product of those of the former</u>

two; hence, the circles by pairs are mutually inverse in the same circle.

39. The point-triads 1,2,3 and α,β,γ are in triple perspective as follows, likewise 1',2',3' and α,β,γ,

1	2	3	with Center.
α	γ	β	1'
β	α	γ	2'
γ	β	α	3'

1'	2'	3'	with Center.
α	β	γ	1
γ	α	β	2
β	γ	α	3

For, as is known, two point-triads in triple perspective with triad of the centres of perspection are a set of triads two and two in triple perspection with the points of the third triad as centres of perspection.

Triangles 1,2,3, and A,B,C.

We shall speak of two triangles in perspective as triangles 1,2,3 and 1'2'3', viz.,

1	2	3
1'	2'	3'
2'	3'	1'
3'	1'	2'

or

1'	2'	3'
1	2	3
3	1	2
2	3	1

,

as being cyclically in triple perspective. When the order of the sides is different we shall say anticyclic.

Then triangles 123 and ABC as point-triads

will be shown to be in anti-cyclic triple perspective with centers α', β', γ'. From coördinates (36) and (39) we write down the coördinates of the rays,—

$$\overline{1\mathfrak{A}}: \ \frac{a-t^3-(1-a^2)t}{a-t^3}, \ \frac{1-at^3+(1-a^2)t^2}{at^3-1}, \ \frac{t(1-a^2)(t^4+at^3-at-1)}{(a-t^3)(at^3-1)}.$$

$$\overline{2\mathfrak{C}}: \ \frac{\omega^2(a-t^3)-\omega(1-a^2)t}{a-t^3}, \ \frac{\omega(1-at^3)+\omega^2(1-a^2)t^2}{at^3-1}, \ \frac{\omega t(1-a^2)(t^4+\omega a t^3-at-\omega)}{(a-t^3)(at^3-1)}.$$

$$\overline{3\mathfrak{B}}: \ \frac{\omega(a-t^3)-\omega^2(1-a^2)t}{a-t^3}, \ \frac{\omega^2(1-at^3)+\omega(1-a^2)t^2}{at^3-1}, \ \frac{\omega^2 t(1-a^2)(t^4+\omega^2 a t^3-at-\omega^2)}{(a-t^3)(at^3-1)}.$$

The determinant of these coördinates vanishes, hence the three lines meet in a point,

$$\alpha' \equiv x = \frac{(1-a^2)t}{at^3-1} \cdot \frac{\begin{vmatrix} (1-a^2)t^2 & at^3-1 \\ at^3-1 & t(a-t^3) \end{vmatrix}}{\begin{vmatrix} a-t^3 & (1-a^2)t^2 \\ (1-a^2)t & at^3-1 \end{vmatrix}} = \frac{(1-a^2)t}{a(at^3-1)}.$$

Similarly, we find, $\beta' = \dfrac{\omega^2(1-a^2)t}{a(at^3-1)}$, and $\gamma' = \dfrac{\omega(1-a^2)t}{a(at^3-1)}$.

The radius of the circumcircle of $\alpha'\beta'\gamma'$ is $\dfrac{1-a^2}{a}\sqrt{\dfrac{t^3}{(at^3-1)(a-t^3)}}$ (41)

The axes of perspective are found by taking the intersections of the sides thus,

1	2	3	
A	Γ	𝔅	giving side of Δ 1′2′3′ opposite 1′.
𝔅	A	Γ	" " " " " " 2′.
Γ	𝔅	A	" " " " " " 3′.

By the same steps, we show that

<u>Triangles 1′2′3′ and ABC</u>

are in cyclic triple perspective with centres of perspective $\alpha'', \beta'', \gamma''$ and axes the sides of triangle 123 in order 1, 3, 2.

The radius of the circumcircle of $\alpha'' \beta'' \gamma''$

is $$a(1-a^2)\sqrt{\frac{t^3}{(at^3-1)(a-t^3)}}. \qquad (42)$$

From a comparison of the radii of $\alpha\beta\gamma$ (35′), $\alpha'\beta'\gamma'$ (41), and $\alpha''\beta''\gamma''$ (42), we may summarize thus

<u>Circles of $\alpha\beta\gamma$, the perspective centers of $\left\{\begin{smallmatrix}1 & 2 & 3\\ 1' & 2' & 3'\end{smallmatrix}\right\}$, and</u>

(1) <u>$\alpha'\beta'\gamma'$, the centers of $\left\{\begin{smallmatrix}1 & 2 & 3\\ A & C & B\end{smallmatrix}\right\}$ whose axes are 1′, 2′, 3′, are inverse to circle of 1, 2, 3.</u>

(2) <u>$\alpha''\beta''\gamma''$, the centers of $\left\{\begin{smallmatrix}1' & 2' & 3'\\ A & B & C\end{smallmatrix}\right\}$ whose axes are 1, 3, 2, are inverse to circle of 1′, 2′, 3′.</u>

See the accompanying figure 4, for the relations found thus far in this chapter.

40. As a more interesting source of triply perspective triangles, and one involving some consideration of the cubic, we [illegible]

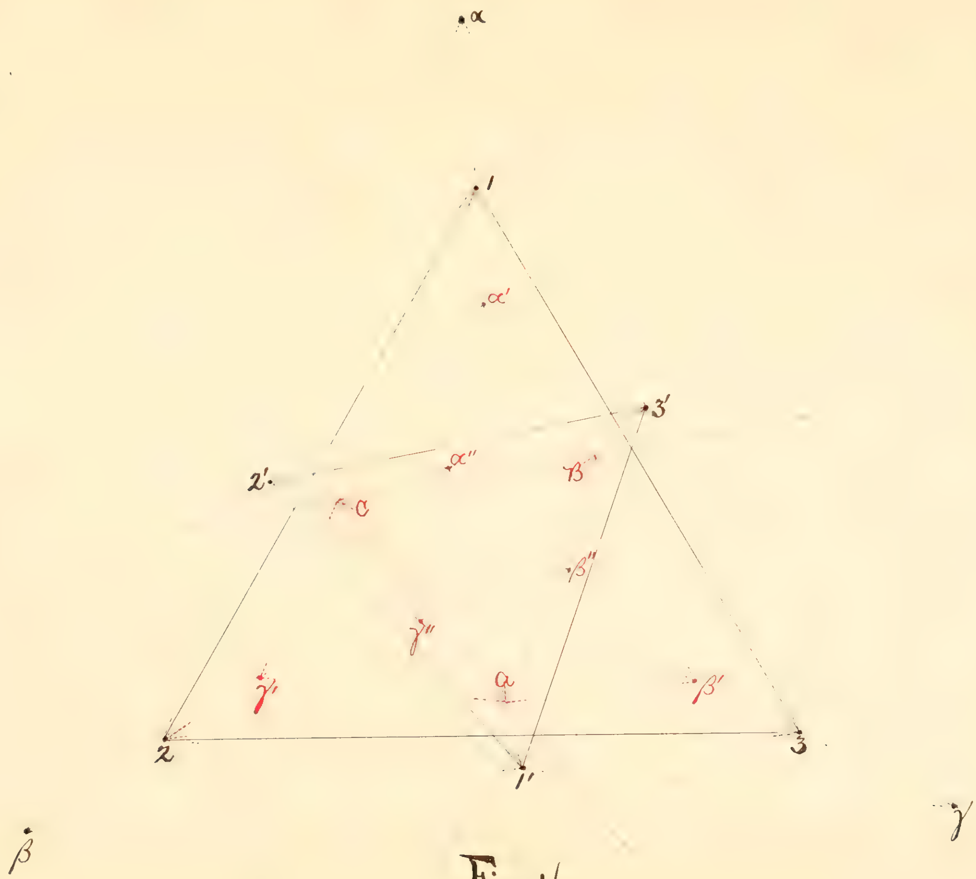

Fig. 4.

The Complete Pappus Hexagon.

This is the hexagon whose sides lie in pairs on two lines, when these have been taken in all possible arrangements. It arises from the consideration of an example in Salmon (1): "If ABC are three points of one line and A'B'C' are three points of another line, then the intersections BC'/B'C, CA'/C'A, AB'/A'B lie on a line". I have not been able to see the works of Pappus to learn his statement of the problem. The most important mention of the simple case is by Rudolf Böger (2), who gives it as a simple form, free from the idea of projective relations, of "Das Sechseck in der Geometrie der Lage".

The complete figure is constructed thus;—

Given two right lines L_1 and L_2 on which are points π_1 π_3 π_5 and π_2 π_4 π_6 respectively. The cross-joins of these points in the six possible ways

(1) Salmon-Fiedler: (6) Aufl., II. T., §285, B. I, s. 466.
(2) Sechseck und Involution, Mittheilungen d. Math. Ges. in Hamburg, Band III., Heft 9, Feb. 1899, s. 3?7.

intersect in points which lie by threes on six lines P_i, which lie by threes on each of two points Σ_1, Σ_2.

To indicate how the points are joined we write the following scheme, which means for example that the lines of (1,2) and (4,5), of (2,3) and (5,6), of (3,4) and (6,1) intersect in three points which lie on the line P_1, and that this line lies on the point Σ_1:–

By permuting **even** numbers	1	2	3	4	5	6	giving	P_1	on Σ_1.
	1	6	3	2	5	4	"	P_2	
	1	4	3	6	5	2	"	P_3	
By permuting **odd** numbers	1	2	5	4	3	6	"	P_4	on Σ_2.
	3	2	1	4	5	6	"	P_5	
	5	2	3	4	1	6	"	P_6	

We name the points of intersection of the nine cross-joins of the six points, π_i, thus,–

The join of π_2 π_4 π_6 to meet the succeeding like join, with resp.,

π_1	π_5	π_3	is called	1	2	3.
π_3	π_1	π_5		4	5	6.
π_5	π_3	π_1		7	8	9.
π_1	π_3	π_5	" "	10	11	12.
π_3	π_5	π_1		13	14	15.
π_5	π_1	π_3		16	17	18.

This means for example that the line $(\pi_2 \pi_1)$ meets the line $(\pi_4 \pi_5)$ in the point 1; $(\pi_4 \pi_5)$ meets $(\pi_6 \pi_3)$ in point 2.

Hence, by comparing this with the previous scheme we see that points

1, 6, 8 lie on P_1.		10, 14, 18 lie on P_4.	
2, 4, 9 " " P_2.		12, 13, 17 " " P_5.	
3, 5, 7 " " P_3.		11, 15, 16 " " P_6.	

The intersections of the lines P_i are named—

Lines P_1	P_2	P_3	meet lines			
P_4	P_5	P_6	resp. in	α_1	α_2	α_3
P_5	P_6	P_4	points	β_3	β_1	β_2
P_6	P_4	P_5	named	γ_2	γ_3	γ_1.

Following the names given to lines and points in the Pascal hexagon we call the points Σ_1 and Σ_2 Steiner points, the lines P_i Pascal lines. The lines L_1 and L_2 we call Hessian diagonals.

41. From the complete Pappus hexagon we have these theorems with their duals:—

I.

Three lines P_i on each of two points Σ_1, Σ_2, joined by the line s, intersect cross-	Three points π_i on each of two lines L_1, L_2, meeting in the point S, are cross-joined

wise in nine points $\alpha_{1-3}, \beta_{1-3}, \gamma_{1-3}$ which join by 18 lines, the sides of two sets of three point-triads each. The triangles of each set are *inter se* in triple perspective, with centers of perspective the points Σ_1 and Σ_2 each three times and also three points on the line D_2 or D_1; with axes of perspective the line D_1 or D_2 three times each for the sets respectively, and twelve other lines all of which lie on the point ϵ, the pole of S as to any of the six triangles. The 18 lines lie by threes on six points π_i on D_1 and D_2.

by nine lines which meet in eighteen points, 1-18, the vertices of two sets of three line-triads each. The triangles of each set are *inter se* in triple perspective, with axes of perspective the lines D_1 and D_2 each three times and also three lines on the point Σ_2 or Σ_1; with centers of perspective the point Σ_1 or Σ_2 three times each for the sets respectively, and twelve other points all of which lie on the line E, the polar of S as to any of the six triangles. The 18 points lie by threes on the six lines P_i on Σ_1 and Σ_2.

Note. The article *the* is omitted in the last sen-

tence of the theorem on the left because in general the six points are not the same points π_i of the right theorem, but are six points with the same Hessian pair for the three on a line.

II.

The lines L_1, L_2, and S are the false sides of the complete quadrilateral of the Hessian pairs of the line-triads, P_i, on Σ_1 and Σ_2 resp.

The points Σ_1, Σ_2 and δ are the false vertices of the complete quadrangle of the Hessian pairs of the point-triads, π_i on L_1 and L_2 resp.

From these follows the general theorem,

III.

Three lines on each of two points give rise to three points on each of two lines, and conversely. They also determine a point and a line invariant for all triads of lines and points with the same Hessian pair respectively.

42. We now give the proof of the theorems on left.

Take the two Steiner points, Σ_1, Σ_2, to have coördinates $\sigma_1, \sigma_2, \sigma_3$ and s_1, s_2, s_3 respectively; the triangle $\alpha_1, \alpha_2, \alpha_3$, formed by the intersections of the Pascal lines as previously indicated, § p. , as the reference triangle; also the line J as auxiliary line.

The line J determined by Σ_1 and Σ_2 is given by the equation $J_1 x_1 + J_2 x_2 + J_3 x_3 = 0$, where $J_i = \begin{vmatrix} \sigma_j & \sigma_k \\ s_j & s_k \end{vmatrix}$. Since this line is taken as auxiliary line, $J_1 = J_2 = J_3 = 1$; $\sigma_1 + \sigma_2 + \sigma_3 = 0 = s_1 + s_2 + s_3$; also there results the relation

$$s_i^2 \sigma_j \sigma_k - s_j s_k \sigma_i^2 = \sigma_k s_k - \sigma_j s_j .$$

The equations of the Pascal lines are the corresponding minors as represented thus,

$$\begin{array}{ccc} P_1 & P_2 & P_3 \\ \hline \sigma_1 & \sigma_2 & \sigma_3 \\ x_1 & x_2 & x_3 \end{array} \qquad \begin{array}{ccc} P_4 & P_5 & P_6 \\ \hline s_1 & s_2 & s_3 \\ x_1 & x_2 & x_3 \end{array}$$

Their respective intersections, $\binom{6}{2} - 6 = 9$ since there are three at each point Σ, have coördinates,—

$\alpha_1 : (1, \; 0, \; 0)$. $\alpha_2 : (0, \; 1, \; 0)$. $\alpha_3 : (0, \; 0, \; 1)$.

$\beta_1 : (\sigma_1 s_1, \; \sigma_1 s_2, \; \sigma_3 s_1)$. $\beta_2 : (\sigma_1 s_2, \; \sigma_2 s_2, \; \sigma_2 s_3)$. $\beta_3 : (\sigma_3 s_1, \; \sigma_2 s_3, \; \sigma_3 s_3)$.

$\gamma_1 : (\sigma_1 s_1, \; \sigma_2 s_1, \; \sigma_1 s_3)$. $\gamma_2 : (\sigma_2 s_1, \; \sigma_2 s_2, \; \sigma_3 s_2)$. $\gamma_3 : (\sigma_1 s_3, \; \sigma_3 s_2, \; \sigma_3 s_3)$.

From these we write the coördinates of the joins,—

$\alpha_2\alpha_3 : (1, \ 0, \ 0)$. $\alpha_3\alpha_1 : (0, \ 1, \ 0)$. $\alpha_1\alpha_2 : (0, \ 0, \ 1)$.

$\beta_2\beta_3 : (\sigma_2 s_3, \ \sigma_3 s_3, \ \sigma_2 s_2)$. $\beta_3\beta_1 : (\sigma_3 s_3, \ \sigma_3 s_1, \ \sigma_1 s_1)$. $\beta_1\beta_2 : (\sigma_2 s_2, \ \sigma_1 s_1, \ \sigma_1 s_2)$.

$\gamma_2\gamma_3 : (\sigma_3 s_2, \ \sigma_3 s_3, \ \sigma_2 s_2)$. $\gamma_3\gamma_1 : (\sigma_3 s_3, \ \sigma_1 s_3, \ \sigma_1 s_1)$. $\gamma_1\gamma_2 : (\sigma_2 s_2, \ \sigma_1 s_1, \ \sigma_2 s_1)$.

By observation we see that these lines three of a column lie on a point. The three points are called π_1, π_3, π_5 respectively with coördinates,

$(0, -\sigma_2 s_2, \ \sigma_3 s_3)$, $(\sigma_1 s_1, \ 0, -\sigma_3 s_3)$, $(-\sigma_1 s_1, \ \sigma_2 s_2, \ 0)$;

and these three points are at once seen to lie on a line, called <u>the line L_{11}</u>, with coördinates

$(\sigma_2\sigma_3 s_2 s_3, \ \sigma_3\sigma_1 s_3 s_1, \ \sigma_1\sigma_2 s_1 s_2)$ or $\left(\frac{1}{\sigma_1 s_1}, \frac{1}{\sigma_2 s_2}, \frac{1}{\sigma_3 s_3}\right)$.

Again, from the coördinates at the end of p. we write the coördinates of the other set of joins,—

$\alpha_1\beta_1 : (0, -\sigma_3 s_1, \ \sigma_1 s_2)$. $\beta_1\gamma_1 : (\sigma_3 s_3 - \sigma_2 s_2, \ -\sigma_1 s_1, \ \sigma_1 s_1)$. $\gamma_1\alpha_1 : (0, -\sigma_1 s_3, \ \sigma_2 s_1)$.

$\alpha_2\beta_2 : (\sigma_2 s_3, \ 0, -\sigma_1 s_2)$. $\beta_2\gamma_2 : (\sigma_2 s_2, \ \sigma_1 s_1 - \sigma_3 s_3, \ -\sigma_2 s_2)$. $\gamma_2\alpha_2 : (\sigma_3 s_2, \ 0, -\sigma_2 s_1)$.

$\alpha_3\beta_3 : (-\sigma_2 s_3, \ \sigma_3 s_1, \ 0)$. $\beta_3\gamma_3 : (-\sigma_3 s_3, \ \sigma_3 s_3, \ \sigma_2 s_2 - \sigma_1 s_1)$. $\gamma_3\alpha_3 : (-\sigma_3 s_2, \ \sigma_1 s_3, \ 0)$.

The three lines here of each column are clearly on a point. These three points we call π_2, π_4, π_6 and their respective coördinates are found to be

$\pi_2 : (\sigma_3 s_2, \sigma_1 s_3, \sigma_2 s_1)$, $\pi_4 : (\sigma_2 s_2 + \sigma_3 s_3 - \sigma_1 s_1, \sigma_3 s_3 + \sigma_1 s_1 - \sigma_2 s_2, \sigma_1 s_1 + \sigma_2 s_2 - \sigma_3 s_3)$,

$$\pi_6 : (\sigma_2 s_3, \sigma_3 s_1, \sigma_1 s_2).$$

The determinant of these is seen to vanish by adding columns; hence, the three points lie on a line, the line δ_2, whose coördinates are

$$\frac{\sigma_2 s_2 - \sigma_3 s_3}{\sigma_1 s_1}, \quad \frac{\sigma_3 s_3 - \sigma_1 s_1}{\sigma_2 s_2}, \quad \frac{\sigma_1 s_1 - \sigma_2 s_2}{\sigma_3 s_3}.$$

The intersection of δ_1 and δ_2, viz., the point δ, has coördinates,

$$\sigma_1 s_1 (\sigma_2 s_2 + \sigma_3 s_3 - 2\sigma_1 s_1), \quad \sigma_2 s_2 (\sigma_3 s_3 + \sigma_1 s_1 - 2\sigma_2 s_2), \quad \sigma_3 s_3 (\sigma_1 s_1 + \sigma_2 s_2 - 2\sigma_3 s_3).$$

The nine joins of each of the two sets, whose coördinates are given on the previous page, are the sides of three triangles. These are point-triads,

(1)	α_1 α_2 α_3		(1)	α_1 β_1 γ_1
(2)	β_1 β_2 β_3	and	(2)	α_2 β_2 γ_2
(3)	γ_1 γ_2 γ_3		(3)	α_3 β_3 γ_3 ,

whose vertices as shown on p. are, for each triangle, joins two and two of all six lines on $\Sigma_1 \Sigma_2$.

The triangles of each set are mutually in triple perspective. The three centers are point π_2, π_4, π_6, as

the coördinates of the above lines given respectively, and in each case the points Σ_1 and Σ_2 as the naming of the points on β shows. The centers for the second set are π_1, π_3, π_5 and Σ_1 and Σ_2 each three times.

The axes of perspective are clearly, one for each two triangles of first set the line h_1, and one for each two triangles of second set the line h_2. This accounts for six of the eighteen axes. The remaining twelve are calculated thus

$\alpha_1\alpha_2$ with $\beta_2\beta_3$ meets in point $(\sigma_3, -\sigma_2, 0)$. $\therefore$ Axis I. has
$\alpha_2\alpha_3$ " $\beta_3\beta_1$ " " " $(0, \sigma_1, -\sigma_3)$. coordinates
$\alpha_3\alpha_1$ " $\beta_1\beta_2$ " " " $(-\sigma_1, 0, \sigma_2)$. $(\sigma_2, \sigma_3, \sigma_1)$,

for by ~~subtracting~~ rows 2nd to first and 3rd to 2nd we have $\sigma_3\sigma_1\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -\sigma_1 & 0 & \sigma_2 \end{vmatrix} = 0.$

$\alpha_1\alpha_2$ with $\beta_3\beta_1$ meets in point $(s_1, -s_3, 0)$. whence axis II. has
$\alpha_2\alpha_3$ " $\beta_1\beta_2$ " " " $(0, s_2, -s_1)$. coördinates
$\alpha_3\alpha_1$ " $\beta_2\beta_3$ " " " $(-s_2, 0, s_3)$. (s_3, s_1, s_2).

Thus in order as indicated we find,—

Triangles.	Centers.	Axes of Perspective.
$\alpha_1\ \alpha_2\ \alpha_3$ / $\beta_1\ \beta_2\ \beta_3$	$\pi_2, \Sigma_2, \Sigma_1$.	$\mathcal{D}_1$, $(\sigma_2, \sigma_3, \sigma_1)$, (s_3, s_1, s_2).
$\alpha_1\ \alpha_2\ \alpha_3$ / $\gamma_1\ \gamma_2\ \gamma_3$	$\pi_4\, \Sigma_1, \Sigma_2$.	$\mathcal{D}_1$, (s_2, s_3, s_1), $(\sigma_3, \sigma_1, \sigma_2)$.
$\beta_1\ \beta_2\ \beta_3$ / $\gamma_1\ \gamma_2\ \gamma_3$	$\pi_6\, \Sigma_2, \Sigma_1$.	$\mathcal{D}_1$, $(\sigma_1, \sigma_2, \sigma_3)$, (s_1, s_2, s_3).

FOLD OUT

Figs. 6, 7, 8.

Triangles.	Centers.	Axes of Perspective.
$\alpha_1\ \beta_1\ \gamma_1$ / $\alpha_2\ \beta_2\ \gamma_2$	$\pi_1, \Sigma_2, \Sigma_1$.	D_2, $(\sigma_2, \sigma_1, \sigma_3)$, (s_2, s_1, s_3).
$\alpha_2\ \beta_2\ \gamma_2$ / $\alpha_3\ \beta_3\ \gamma_3$	$\pi_3, \Sigma_2, \Sigma_1$.	D_2, (s_3, s_2, s_1), $(\sigma_3, \sigma_2, \sigma_1)$.
$\alpha_3\ \beta_3\ \gamma_3$ / $\alpha_1\ \beta_1\ \gamma_1$	$\pi_5, \Sigma_1, \Sigma_2$.	D_2, $(\sigma_1, \sigma_3, \sigma_2)$, (s_1, s_3, s_2).

The twelve axes other than D_1, D_2 evidently pass through a point with coördinates $(1, 1, 1)$, ε, which makes it the auxiliary point, or the pole of T as to the reference triangle. Since

S has the same pole, ε, as to all point-cubics consisting of three joins of the six lines P_i by twos,	S has the same polar, E, as to all line-cubics consisting of three joins of the six points π_i by twos,
ε is the pole of T as to every and any of the six triangles on all six lines P_i.	E is the polar of S as to every and any of the six triangles on all six points π_i.

The 18 lines have already been shown, on p. 83, to lie by threes on the six points π_i which lie on the lines D_1 and D_2 three and three.

(1) Salmon: H. P. C. § 166, pp. 143, 144.

The lines L_1 and L_2 as diagonals or false sides of the complete quadrilateral of the Hessian pairs.

The following linear relation exists between the three lines, P_1 P_2 P_3, on Σ_1,–

$$\sigma_1(\sigma_2 x_3 - \sigma_3 x_2) + \sigma_2(\sigma_3 x_1 - \sigma_1 x_3) + \sigma_3(\sigma_1 x_2 - \sigma_2 x_1) = 0.$$

The Hessian covariant of the binary cubic is given by the sum of the squares of these terms separately; and each of the imaginary Hessian lines separately is

$$\sigma_1(\sigma_2 x_3 - \sigma_3 x_2) + \omega\sigma_2(\sigma_3 x_1 - \sigma_1 x_3) + \omega^2\sigma_3(\sigma_1 x_2 - \sigma_2 x_1) = 0$$

and $$\sigma_1(\sigma_2 x_3 - \sigma_3 x_2) + \omega^2\sigma_2(\sigma_3 x_1 - \sigma_1 x_3) + \omega\sigma_3(\sigma_1 x_2 - \sigma_2 x_1) = 0.$$

These two and the analogous pair on Σ_2 may be written in reduced form respectively

$$\sigma_2\sigma_3 x_1 + \omega\sigma_3\sigma_1 x_2 + \omega^2\sigma_1\sigma_2 x_3 = 0, \qquad [1]$$

$$\sigma_2\sigma_3 x_1 + \omega^2\sigma_3\sigma_1 x_2 + \omega\sigma_1\sigma_2 x_3 = 0, \qquad [2]$$

$$s_2 s_3 x_1 + \omega s_3 s_1 x_2 + \omega^2 s_1 s_2 x_3 = 0, \qquad [3]$$

$$s_2 s_3 x_1 + \omega^2 s_3 s_1 x_2 + \omega s_1 s_2 x_3 = 0. \qquad [4]$$

These intersect as follows,

[1] and [3] in the imaginary point **I**: $(\sigma_1 s_1,\ \omega^2\sigma_2 s_2,\ \omega\sigma_3 s_3)$.

[2] " [1] " " " " **J**: $(\sigma_1 s_1,\ \omega\sigma_2 s_2,\ \omega^2\sigma_3 s_3)$.

[1] " [1] " " " " **H**: $\big(\sigma_1 s_1(\sigma_2 s_2-\omega\sigma_3 s_3),\ \sigma_2 s_2(\sigma_3 s_3-\omega\sigma_1 s_1),\ \sigma_3 s_3(\sigma_1 s_1-\omega\sigma_2 s_2)\big)$.

[2] " [3] " " " " **K**: $\big(\sigma_1 s_1(\omega\sigma_2 s_2-\sigma_3 s_3),\ \sigma_2 s_2(\omega\sigma_3 s_3-\sigma_1 s_1),\ \sigma_3 s_3(\omega\sigma_1 s_1-\sigma_2 s_2)\big)$.

The line of $\overline{IJ}$ is the <u>real</u> line λ_1, given on p. 83.

" " " $\overline{HK}$ " " " " λ_2 " " " 84.

Therefore, λ_1 and λ_2 are the diagonals of the imaginary quadrilateral of the Hessian pairs of the line-triads on Σ_1 and Σ_2, respectively.

Theorem III. follows without further proof.

44. As may be seen from Fig. 5, the three line-triads of a set are mutually in triple perspective thus:

Triangle.	Centers.	Axes of Perspective.
1, 2, 3 / 4, 5, 6	a, b, Σ_1.	Δ_2, Δ_1, P_5.
1, 2, 3 / 7, 8, 9	c, Σ_1, d.	$\lambda_2, P_6, \lambda_1$.
4, 5, 6 / 7, 8, 9	e, f, Σ_1.	Δ_2, Δ_1, P_4.
10, 11, 12 / 13, 14, 15	g, Σ_2, h.	Δ_2, P_1, Δ_1.
10, 11, 12 / 16, 17, 18	k, l, Σ_2.	P_3, Δ_1, Δ_2.
13, 14, 15 / 16, 17, 18	m, Σ_2, n.	Δ_2, P_2, Δ_1.

Without formal proof, because they follow analytically from data that have been given or may be tested in figures 5 and 6, we present several theorems,—

IV.

A triangle of one set of three is in two-fold perspective with any one of the associated set of three, but

there are only nine axes each taken twice and situated on the invariant point ε. The centers in each case are Σ_2 and Σ_1.

there are only nine centers each taken twice and situated upon the invariant line E. The axes in each case are D_2 and D_1.

V.

The point- and line-triads are between themselves in single perspective. The center of perspective in each case is Σ_1 if the two triads are both of the first or both of the second set of three as they were associated, and Σ_2 if of oppositely named sets. (The axes are not in general D_1 and D_2, since the duality breaks hold.— See remark following Th. I.)

45. The theorems thus far have not in the least discriminated the lines of the points, it will be noticed. This fact suggests enquiry as to special cases.

The special forms for the three lines on each of the two points to which attention is called are,

(1) Two sets of equispaced triads; i.e., lines at 120°.

(a) the points being the equiangular points of the triangles of one set of three.

(b) the points being on the circumcircle of equiangular triads, the 3 of either set.

(2) the points taken at infinity

(a) arbitrarily, giving two sets of 3 parallel lines each at an angle with those of the other set.

(b) at I and J, giving two sets of perpendicular lines.

These cases suggest special methods of analysis, two of which were given by Prof. Morley this year in lectures. The methods even apply to forms

(1)-(a), and (2)-(b). I find that the method by circular coördinates is well adapted to a further specialized form of (1)-(a), and shall presently prove the Prop in this way.

46. First, I would state that Fig. 5 was not completed to show fully both dual parts of Theorem I. as one of the three lines on Σ_2 was planned to pass through Σ_1. It is difficult to set a figure to show the whole matter upon a paper of limited size. Fig. 7 starts with the 3 lines on each of two points and can be carried on to show both parts but was left as it is, to avoid having it too complicated. Fig. 8 shows the case of triads on I and J.

In these special forms of the points and lines we have some special theorems:

VI.

For a triad of equispaced lines on each of two points, one set of the three point-triads of the intersections of these lines two and two is perpendicular

triangles with sides respectively parallel [Fig. 6]; thus one of the Hessian diagonals, ε, is at infinity; the other is the perpendicular bisector of the line s between Σ_1 and Σ_2. [illegible] the circumcircles of the 3 equilateral triads pass through Σ_1 and Σ_2, and those of the other set of three triangles intersect in ε, the pole of the line s as to any of the six triangles.

VII.

If one of the triangles of the set of acute triangles is also equilateral, the [illegible] is that the triads of the set are inverse points with respect to the circumcircle of this triangle, and all the circumcenters of the 3 equilateral triangles are on the finite Hessian diagonal.

We give this last using circular coördinates.

47. Call Σ_1 the origin, $\Sigma_2 \equiv t$, on the circumcircle of the extra equilateral triangle as unit circle with Σ_1 as its center; also call the vertices of this triangle,

$\alpha_1 \equiv 1, \quad \alpha_2 \equiv \omega, \quad \alpha_3 \equiv \omega^2.$

The coördinates of the Pascal lines are at once written,–

$P_1 : (1, -1, 0). \quad P_2 : (1, -\omega^2, 0). \quad P_3 : (1, -\omega, 0).$

$P_4 : (1, t, -t-1). \quad P_5 : (1, \omega t, -t-\omega). \quad P_6 : (1, \omega^2 t, -t-\omega^2).$

Their nine intersections are,–

$\alpha_1 : 1. \quad \beta_1 : \frac{t+\omega^2}{t+1}. \quad \gamma_1 : \frac{t+\omega}{t+1}.$

$\alpha_2 : \omega. \quad \beta_2 : \frac{t+1}{\omega^2 t+1}. \quad \gamma_2 : \frac{t+\omega^2}{\omega^2 t+1}.$

$\alpha_3 : \omega^2. \quad \beta_3 : \frac{t+\omega}{\omega t+1}. \quad \gamma_3 : \frac{t+1}{\omega t+1}.$

Since each is the reciprocal of the inverse of the other, respectively, as observation shows,

β_1 and γ_3, β_2 " γ_1, β_3 " γ_2 } are inverse points as to the unit circle about $\alpha_1 \alpha_2 \alpha_3$.

The coördinates of the sides are easily found:

$\alpha_1\alpha_2 : (1, \omega, \omega^2). \quad \alpha_2\alpha_3 : (1, 1, 1). \quad \alpha_3\alpha_1 : (1, \omega^2, \omega).$

$\beta_1\beta_2 : \quad t^2-\omega^2 t+1, \quad t^2-\omega t+1, \quad \omega^2(t+\omega^2)(t+1).$

$\beta_2\beta_3 : \quad t^2-t+\omega^2, \quad \omega^2 t^2-\omega t+\omega, \quad (t+\omega)(t+1).$

$\beta_3\beta_1 : \quad \omega^2 t^2-t+1, \quad t^2-t+\omega, \quad (t+\omega^2)(t+\omega).$

$\gamma_1\gamma_2 : \quad \omega t^2-t+1, \quad t^2-t+\omega^2, \quad (t+\omega)(t+\omega^2).$

$\gamma_2\gamma_3 : \quad t^2-t+\omega, \quad \omega t^2-\omega^2 t+\omega^2, \quad (t+\omega^2)(t+1).$

$\gamma_3\gamma_1 : \quad t^2-\omega t+1, \quad t^2-\omega^2 t+1, \quad \omega(t+1)(t+\omega).$

$\alpha_1\beta_1$: $\quad t, \quad \omega^2, \quad -(t+\omega^2).$

$\beta_1\gamma_1$: $\quad t(t+1), \quad t+1, \quad -(t+\omega^2)(t+\omega).$

$\gamma_1\alpha_1$: $\quad t, \quad \omega, \quad -(\omega t+\omega).$

$\alpha_2\beta_2$: $\quad t, \quad \omega^2, \quad -\omega(t+1).$

$\beta_2\gamma_2$: $\quad t(t+\omega), \quad t+\omega, \quad -\omega(t+1)(t+\omega^2).$

$\gamma_2\alpha_2$: $\quad t, \quad \omega, \quad -\omega(t+\omega^2).$

$\alpha_3\beta_3$: $\quad t, \quad \omega^2, \quad -(\omega^2 t+1).$

$\beta_3\gamma_3$: $\quad t(\omega t+1), \quad \omega t+1, \quad -(t+1)(t+\omega).$

$\gamma_3\alpha_3$: $\quad t, \quad \omega, \quad -(\omega^2 t+\omega^2).$

The above three are evidently equiangular with corresponding sides parallel, as the coördinates show.

As shown for the general theorem, the sides meet by twos with three of these, meet on a line, which is one of the axes. The coördinates of these axes are calculated as shown before in using circular coördinates and are found to be,–

I. :	$t^4,$	$\omega^2,$	$\omega t^2.$
II. :	$t^2,$	$-\omega t,$	$\omega t^3 - 1.$
III. :	$t^2,$	$-\omega^2 t,$	$\omega^2 t^3 - 1.$
IV. :	$t^4,$	$\omega,$	$\omega^2 t^2.$
V. :	$t^4(t^3-2),$	$-(2t^3-1),$	$t^2(t^3+1).$
VI. :	$t^2(2t^3-1),$	$t(t^3-2),$	$-(t^6-1).$
VII. :	$t^3,$	$\omega,$	$-\omega t(t+\omega^2).$
VIII. :	$t^2,$	$-\omega^2,$	$\omega^2(t^2-\omega).$

Etc., etc.

These all clearly meet in the point t^{-2}, which is ε, and is on the unit circle at a point twice as far from the unit-point along the circle as t and in the opposite direction.

Thus as we vary the location of Σ_2 on the circle of $\alpha_1 \alpha_2 \alpha_3$, the point ε varies twice as rapidly in the opposite direction. They thus coincide in three points on the circle.

Vita.

Charles Clayton Grove was born December 19th, 1875, the son of Lewis and J. Elizabeth Grove, at Hanover, Pa. His early education was received in the private school of Miss Martha C. Grove and in the public schools of Hanover.

In 1896 he graduated from the Millersville State Normal School, then he, after three years of teaching, he entered in September, 1898 upon the Classical Course of Pennsylvania College, Gettysburg, receiving the degree of A.B. in 1900. The next winter he was Supervising Principal of the schools of Hummelstown, Pa. In 1903 the degree of A.M. was conferred at Gettysburg.

In October 1901 he entered upon graduate work at Johns Hopkins University with Mathematics, Physics and Italian as subjects. At its very close this course was interrupted by his taking an instructorship in Mathematics at State College, Pa., for spring of 1905.